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COMPARATIVE ANALYSIS OF OPTIMAL MAINTENANCE POLICIES UNDER GENERAL REPAIR WITH UNDERLYING WEIBULL DISTRIBUTIONS

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Abstract – Various replacement policies under Kijima's general repair model with the underlying Weibull distribution function are studied via two efficient methods. The first one is based on our previously derived approximate formula for the g renewal function; the second is an improved Monte Carlo method. These methods enable an in–depth, comparative analysis of the maintenance polices in question. An efficient algorithm is suggested for finding optimal preventive replacement times. The influence of restoration factor, including the deviation from a minimal repair assumption, on the optimal solution is studied. A practical study illustration is provided.

Index Terms – G renewal process, optimal maintenance, Weibull distribution, Monte Carlo method.

NOTATION

V_n, S_n – virtual, real age of the system after repair

n q – restoration (or repair effectiveness) factor

t – time

W(t) – g renewal function denoting the expected cumulative number of failures

f(t) – probability density function

F(t) – cumulative distribution function

λ, α – respectively, the scale, and the shape parameters of Weibull distribution

Γ(x) – Gamma function

μ, σ – the mean, and the standard deviation of the failure time distribution

N – number of simulations

a, b, c, , A, B, H, D – numerical constants

C_0 – replacement cost

C_1 – minimal repair cost

C_q – corrective repair cost depending on restoration factor

C_T – expected total cost per unit time

T – expected length of replacement cycle

INTRODUCTION

Even though maintenance optimization is of great practical importance, the vast majority of papers [1], [2], [3], [4], [7] were devoted only to two special cases of the repair model: 1) minimal repair, when a system is "same−as−old" following the repair, and 2) perfect repair, when the system is "good−as−new" after restoration. The generalized renewal process (GRP) was introduced in [11], [12], [13], and then developed in [5], [6]. In [12], [13] it was applied to the maintenance optimization problem with special underlying distribution functions including the Gamma function, and some combination of exponential functions, because they imply an easy way to obtain the solution. The most popular Weibull distribution function is considered in a few papers [5], [6]. We did not find, however, a comprehensive, systematic analysis of maintenance policies under the GRP model with the underlying Weibull distribution function. We believe that the main reason is the complexity of the problem: it is required to solve complex g−renewal equations many times to obtain an optimal solution in a general case. We suggest two efficient methods for solving this problem, and provide detailed analysis of maintenance solutions with respect to different GRP parameters and maintenance policies.

The objective of this research was trifold: 1) to demonstrate the efficiency of the Approximate and Improved Monte Carlo methods in solving the g–renewal equations with application to optimal maintenance problems; 2) to study the sensitivity of the g–renewal model (in particular, the sensitivity of the minimal and perfect repair assumptions) to the value of the restoration factor in maintenance optimization; and 3) to comparatively analyze optimization policies under the probabilistic framework of the *g–renewal process* with an underlying Weibull distribution function.

A. Maintenance policies description and assumptions.

We consider a system deteriorating with age in the infinite time horizon. It can be repaired according to the GRP model with cost C_q , or replaced by a new one with cost C_0 . Both types of maintenance can be corrective (at the failure time), or preventive (at a scheduled time), creating four possible combinations.

To prevent the entire degradation of the system in the infinite time horizon, we have to introduce periodic replacement in the maintenance process to set the age of the system to 0 at the end of each cycle. The optimization criterion is the expected average total cost per unit time

$$C_T \qquad E\,\frac{\cos t\ of\ \ cycle}{length\ of\ \ cycle} \tag{1}$$

The total cost C_T can be minimized by scheduling preventive maintenance and selecting different maintenance policies, described for example in [1]. Following [2], [3], we consider 3 types of polices, which were introduced first for the case of minimal repair.

Policy 1 *: Perform repairs up to age T , and replace at age T* [2].

In this case, the length of a cycle is constant, and the expected total cost per unit time can be defined as [3]

$$C_T = \frac{C_0 + C_q W(T)}{T}. \quad (2)$$

According to this policy, replacements are preformed periodically, and it is sufficient to define the renewal process in the time interval $0 \le t \le T$. The objective is to find an optimal value $T = T^*$, which minimizes (2).

Policy 2*: Perform repairs for the first* $n-1$ *failures, and replace at the* n *th failure.*

$$C_T = \frac{C_0 + C_q (n-1)}{T_n}, \quad (3)$$

where $T_n = E[\text{length of cycle}]$. The total cost is minimized with respect to n in this model. This policy is more flexible compared to the previous one, and it yields a lower average total maintenance cost, which is proven in [7] for the minimal repair model. A simple formula and numerical examples are also provided in this case for the underlying Weibull failure time distribution. The policy is not studied in a more general case.

Policy 3*: Perform repairs up to age* T_3^**, and replace at the first failure after* T_3^* [5].

It is shown in [5] that this policy is optimal under a minimal repair model for the given values C_0 and C_1. This result is expanded for the case of the g−renewal process in [4].

In the general case of this policy, (1) should be used. In [11], the authors implicitly assumed that the cost of a cycle and the length of a cycle are *s*−independent, and therefore the following equation is considered

$$C_T = \frac{E[\text{cost of cycle}]}{E[\text{length of cycle}]}. \quad (4)$$

We will show the limitations of this formula using the Monte Carlo method.

Policy 2 is more cost efficient than Policy 1, and Policy 3 is optimal given that both types of cost C_q and C_0 are fixed. In practice, however, the replacement in Policy 1 may be less expensive because it is planned at a given time T, and the unexpected down time can be significantly reduced. We will study this case, and find the limitations when Policies 2 and 3 are actually more efficient compared to the first one.

Obviously, the difference between these three policies depends on all GRP parameters (that is, parameters of the underlying distribution, and the restoration parameter). We did not find any systematic comparative studies of maintenance policies considering the case of the GRP with a Weibull underlying distribution. This analysis will be provided in Section III of the paper.

The average cost of repair C_q (as well as the cost of replacement C_0) includes not only the costs of repair itself but also all costs resulting from the failure (e. g., the cost of down time, possible lost sales, idle labour, delays in logically dependent processes). For example, the expected down time can be calculated as

$$T_d = \int_0^T Q(t)\,dt, \quad (5)$$

where $Q(t)$ is the unavailability of the system at current time point t. If the expected time to repair is much less than the time to failure, $Q(t) \approx w(t)$ [12], where $w(t) = dW(t)/dt$. Finally, the total down time can be represented as $T_d \approx W(T)$. Therefore, the corresponding cost is proportional to $W(T)$, and can be included in (2).

Solving the generalized renewal process equations, and in particular finding the g−renewal function $W(t)$, is the core problem of maintenance optimization.

B. Generalized renewal process.

As discussed in [9] and [13], in repairable system reliability analysis, one could consider four different states to which a system can be repaired following a failure: 1) "good−as−new," 2) "same−as−old," 3) "bet–ter−than−old−but−worse−than−new," and 4) "worse−than−old." All of these states are usually modeled by a *stochastic point processes*. A general assumption, which is made when using a stochastic point process to model a repairable system's failure occurrence, is that the time of the system's repair is negligibly small compared to its time to failure. This assumption is quite realistic in many applications; for example, consider 18 months to an automobile failure vs. 3 days (0.1 months) to its repair.

Upon a failure, if a repairable system is restored to a "good−as−new" condition, and the time between system failures can be treated as an i.i.d. random variable, then the failure occurrence can be modeled by the *Ordinary Renewal Process* (ORP). If upon a failure the system is restored to the "same−as−old" condition, then an appropriate model to describe the failure occurrence can be the *Non−Homogeneous Poisson Process* (NHPP). A more general model is the so−called *Generalized Renewal Process* (GRP), which treats ORP and NHPP as special cases.

The GRP or *g renewal process*, originally introduced by Kijima and Sumita [11], [13], has gained increasing popularity in modeling and analysis of recurrent events, specifically in reliability and maintainability applications. The GRP is introduced (Kijima Model 1) using the notion of *virtual age*:

$$V_n = qS_n. \quad (6)$$

For $q = 0$, the age of the system after the repair is "re−set" to zero; this approach corresponds to the ORP. With $q = 1$, the system is restored to the "same−as−old" condition, which is modeled as a NHPP. The case of $0 < q < 1$ corresponds to the intermediate "bet-ter−than−old−but−worse−than−new" repair assumption.Finally, with $q > 1$, the virtual age is $V_n > S_n$, so that the repair damages (ages) the system to a higher degree than it was just before the respective failure, which corresponds to the "worse−than−old" repair assumption. As such, all four considered cases of q can be modeled by the GRP.

Under the GRP, the expected number of events (failures) in (0, t] is given by a solution of the so−called g renewal function [13]

$$W(t) \quad \int_0^t g(\ |0) \quad \int_0 w(x)g(\ x|x)dx\ d, \quad (7)$$

where

$$g(t\ x) \quad \frac{f(t\ qx)}{1\ F(qx)},\ t, x \quad 0, \textit{and}\ w(x) \quad \frac{dW(x)}{dx};$$

Note that $g(t|0) = f(t)$.

In this paper, we consider the most popular Weibull distribution with the cumulative distribution function (CDF) expressed by

$$F(t)\ 1\ e^{(\ t)} \quad (8)$$

in the time interval $t\ 0$. The scale, and shape parame-ters are restricted to the range 0 , and 0 , respec-tively, in general. We assume in addition that 1 , which corresponds to an Increasing Failure Rate (IFR) (degrading system), and preventive maintenance leads to a decrease in the total cost in this case. We also put 1 without loss of generality.

The closed form solution of the g renewal equation does not exist, and numerical solutions are difficult to obtain. It is desirable to obtain the efficient algorithm in a maintenance optimization problem because obtaining the optimal length of the replacement cycle requires solving the g−renewal equation many times over. A comprehensive list of works on solving the ordinary and the g−renewal equations can be found in [13]. In this paper, we consider two types of solutions: an improved Monte Carlo (MC) method, and an approximate formula for the g−renewal function.

CALCULATION METHODS

A. Monte Carlo method.

A MC approach was introduced for solving the g−renewal problem in [9], and applied to the estimation of the expected number of repairs in warranty data analy-sis [10]. This raw simulation is quite time consuming, if obtaining many solutions for different values of process parameters is required, for example in warranty claims forecasts, or maintenance schedule optimization. An improvement of MC methods was suggested in [11], and effectively implemented in fault tree analysis [12]. The main result of this approach can be applied to the maintenance optimization problem as well.

The GRP is represented in [12] as a continuous time semi−Markov chain, whose state space is defined as a set of states of a system between the i−th and ($i+1$)−th failures (i=0, 1, 2, ...).

$$W(t) \sum_{i\ 0} P_{i,i\ 1}(t) \quad (9)$$

where $P_{i,\ i+1}(t)$ is probability of a transition from state i to state $i+1$ at a given time t under the condition that the system is in the i−th state. The simulation procedure is illustrated in Fig 1. To define the expected number of failures at observation time t in each trial, we calculate the sum of the probabilities of the failures at the given time t for each subsequent time S_i immediately after the i−th repair.

The first term of (9) can be written in the simple form

$$P_{0,1}(t)\ F(t). \quad (10)$$

It does not depend on the trial number, its variance is equal to 0, and therefore the suggested approach is much more accurate.

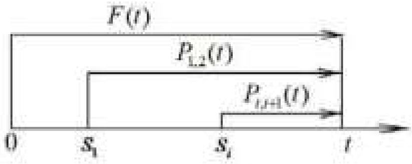

Figure 1 – Simulation procedure according to (9)

The implementation of (9) in the simulation procedure is as follows. For each trial, and at each given time t, we consider all the failures that occurr at time $S_i < t$. We have to put $i > 0$, because $i=0$ corresponds to the first term in (9). Each i−th time to failure t_i at time S_i (and corresponding vertical time V_i) is defined according to (6) and (8) by generating random numbers F_i in the simulation process

$$t_i \quad V_{i1} \quad \ln 1 \quad F_i \quad ^{1/} \quad V_{i1} \quad (11)$$

If necessary, the time to repair also can be included in the MC simulation. In this case, we consider S_i as time immediately after the i−th repair. Then, we calculate the sum (9) of the corresponding probabilities of the transition to the next failure at the given time t

$$W(t)\ F(t) \quad \sum_{S_i\ t} (1\ \ \exp(\quad [t \quad (1\ \ q)S_i \quad qS_i\ ] \quad (12)$$

as it is shown in [12], the maximum required number of MC simulation trials is N=4000 to reach a reasonably high accuracy. It takes about 0.05sec to obtain the result in the interval $0 \quad W\ t \quad 10$ on our computer having 4 processors (2.51 GHz) and 6.0GB of RAM.

B. Approximate solution for the renewal function.

A simple approximate formula for the renewal function was suggested by the authors in [13]:

$$W(t,\ ,q)\ W_0(t,\ )\ q\ W_1(t,\ )\ W_0(t,\ ) \quad (13)$$

where W_0 $t,$ is the renewal function corresponding to ORP, if q=0; $W_1(t,)$ t is the exact solution corresponding to a minimal repair assumption, if q 1 .

If 1 2 ,

$$\frac{a \quad b^{2}}{1 \quad c} \quad , \tag{14}$$

where a 0,3096; b 0,2846 , and c 0,2909. If 2 ≤ α ≤ 5,

$$G_0 \; 2 \quad A_0 \qquad B_0 \; t \quad 0.5q \quad H_0 \qquad D_0 \; t \; , \tag{15}$$

where G_0 1.112 , A_0 0.2176, B_0 0.2846 , H_0 0.09, and D_0 0.48 .

Function W_0 $t,$ is approximated with good accuracy using a two−point Padé function

$$W_0(t) \quad \frac{a_1(\; t) \; \ldots \; a_n(\; t)^{n} \quad A(\; t)^{n} \; t}{1 \quad B(\; t)^{n}} \tag{16}$$

where B 2 a_n ^{2} / ^{2} ^{2} , A B / . Coefficients a_k are defined by a simple recursive procedure according to

$$A_1 \; 1 \quad A_2 \; 21 A_1 \quad \ldots \quad A_k \; A_{ki} \; A_{k-i} \quad k\,1 \quad i\,1 \tag{17}$$

where k k 1 / k ! , and a_k (1)$^{k \; 1}$ A_k / (k 1) . The optimal number of terms in (16) is defined as n 17 , if 1 .

These formulae were tested against the MC method. The relative error did not exceed 2.6% in intervals 0 W t 5 , 1 5 , and 0 q 1 . The formulae were efficiently applied to the warranty prediction problem. We will use them to define the optimal replacing time in the maintenance optimization problem. We can expect that the accuracy of the calculation of the average total cost (2) will be even higher because of the increased calculation accuracy of W(t); the remaining terms in (2) are deterministic quantities, and have no uncertainty associated with them.

CALCULATION RESULTS

A. Maintenance Policy 1

The result of calculations corresponding to Policy 1 is shown in Tables I−IV for various values of GRP parameters and replacement cost C_0 . We assume corrective repair cost C_q 1 in all cases. These results are obtained by approximate formulae (13)−(17), and the Monte Carlo method, respectively. The number of trials in the improved MC method was 10,000, which guarantees the relative Standard Error (SE) of the method to be less than 0.25%. Values of optimal replacement time T^* and the corresponding expected number of failures $W T^*$ are also provided in Tables I−IV.

Table I – Optimal maintenance cost (C_0 = 5, α = 2)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	4.472	4.325	4.166	3.993	3.588	3.061	2.433
Monte Carlo	4.473	4.319	4.152	3.975	3.568	3.027	2.231
Monte Carlo SE	8.5E−3	8.0E−3	7.5E−3	7.0E−3	6.0E−3	4.7E−3	3.0E−3
Length of cycle, T^*	2.216	2.411	2.517	2.659	3.102	4.254	7.497
Number of failures, $W(T^*)$	4.915	5.407	5.4472	5.573	6.077	7.883	11.72

Table II – Optimal maintenance cost (C_0 = 3, α = 2)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	3.464	3.369	3.267	3.152	2.883	2.525	1.949
Monte Carlo	3.472	3.378	3.274	3.146	2.863	2.504	1.939
Monte Carlo SE	7.3E−3	6.9E−3	6.5E−3	6.3E−3	5.5E−3	4.6E−3	3.0E−3
Length of cycle, T^*	1.719	1.808	1.879	2.091	2.552	3.350	6.026
Number of failures, $W(T^*)$	2.969	3.113	3.156	3.578	4.312	5.397	8.690

Table III – Optimal maintenance cost (C_0 = 5, α = 5)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	5.977	5.798	5.582	5.268	4.549	3.676	−
Monte Carlo	5.976	5.787	5.559	5.262	4.521	3.607	2.389
Monte Carlo SE	5.0E−3	5.0E−3	4.2E−3	4.5E−3	3.7E−3	2.9E−3	1.7E−3
Length of cycle, T^*	1.038	1.129	1.193	1.377	1.726	2.497	5.289
Number of failures, $W(T^*)$	1.200	1.596	1.633	2.244	2.808	4.010	7.636

Table IV – Optimal maintenance cost (C_0 = 3, α = 5)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	3.973	3.898	3.815	3.709	3.276	2.781	−
Monte Carlo	3.972	3.890	3.798	3.685	3.279	2.746	1.983
Monte Carlo SE	2.9E−3	2.8E−3	2.5E−3	2.5E−3	2.7E−3	2.4E−3	1.5E−3
Length of cycle, T^*	0.937	0.992	1.047	1.156	1.487	2.167	4.407
Number of failures, $W(T^*)$	0.718	0.852	0.975	1.271	1.880	2.950	5.744

The case of q=1 corresponds to the popular *minimal repair* model. Analyzing data from Tables I−IV, one can conclude that the model is not sensitive with respect to restoration factor q in the vicinity of q=1. For example, if the factor is changed from 1 to 0.9 (by 10%), the corre-sponding optimal cost decreases approximately by 3%. The sensitivity declines if C_0 decreases. Therefore, the most popular, simple minimal repair model is a good approxima-tion for obtaining conservative estimation of the optimal cost, if the expected restoration factor is close to 1.

Comparing the results obtained by the approximate and MC methods, we can see good agreement in the in-terval of q 0.3 . If q 0.1 , we did not obtain a mini-mum cost value using the approximate formulae because they are valid in the range $W T$ 5 . However, we be-lieve that C_0 C_q , q 0 is not a practical case. If q is small, the g −renewal model is close to a perfect repair. In this case, it is more likely that C_0 C_q because, if q 0 , the perfect repair at failure can be considered as if re-placement happened unexpectedly (not at a scheduled time).

We also considered the sensitivity of the *perfect re-pair* model with respect to restoration factor q for vari-ous values of Weibull shape parameter , and replace-ment cost C_0 C_q . If C_0 is smaller compared to C_q (q 0), and the shape parameter is large, then the mini-mal cost slightly depends on restoration factor q . For example, if C_0 0.5 ,5 , and q 0 , the minimal total cost min C_T 0.934 is reached at time value T^* 0.679 , when the expected number of failures is small $W T^*$ 0.130 , and the renewal process does not depend on q , as one would expect in this situation. We obtained min C_T 0.935 if q 0.2 in this case. A cou-ple of curves represent the other case when C_0 0.8 , 5 in Fig. 2.

Curve *1* corresponds to q 0.2 , and has a minimum of 1.352 at T^* 0.735 , when $W T^*$ 0.194 . Curve *1'* corresponds to perfect repair (q 0).

It has a local minimum very close to the previous one; however, it also has a maximum, and then it slowly decreases with cycle time T . Another couple of curves 2 (q 0.2) and *2'* (q 0) correspond to the case C_0 0.5 , 2 . Curve 2 has a minimum of 1.30 at T^* 1.063 , when $W T^*$ 0.876 . The total cost C_T is always de-creasing over time if q 0 , which means that preventive replacement is not needed in this case. Corrective re-placement is more efficient even though it is more expen-sive.

Figure 2 – Total cost C_T depending on length of replacement cycle T

B. Maintenance Policy 2

The result of the calculations is shown in Tables V−VIII for various parameter values of the maintenance process with imperfect repair, and periodic replacement. According to Policy 2, the "target" number of failures n, when the unit is replaced by a new one, and corresponds to the minimal total cost, is calculated using the MC method. The corresponding values of the expected cycle length T^* of the maintenance are also shown in Tables V−VIII.

Policy 2 turns out to be more efficient compared to the Policy 1. The difference in minimal cost is changing from 7.7% to 18% in our examples, corresponding to the minimal repair model. The difference is decreasing if the restoration factor is decreasing. This comparison is made under the assumption that a periodical *corrective* re-placement after the *n−th* failure in Policy 2 has the same cost as a *scheduled preventive* replacement in Policy 1. However, typically the scheduled maintenance cost is less expensive, and therefore Policy 1 can be more efficient.

We suggest also an approximate solution for Policy 2 based on previous calculations from Tables I−IV. n can be obtained as the closest integer to the expected number of failures $W T^*$, corresponding to optimal solution according to Policy 1. These numbers are represented in Tables V−VIII in separated rows, and denoted as n^* . In most cases, the numbers in the tables coincide with n. When they are different, we calculated the difference be-tween the exact and approximate values of total cost. If 2 , this difference did not exceed 1%. If 5 , there is only one case with different values, when C_0 3 , q 0.7 . The difference in total cost is only 1.8% in this case.

To calculate approximate total cost based on data from Tables I−IV, we can suggest (18) instead of (3):

$$C_T \quad \frac{C_0\ C_q (W T^* \quad 1)}{T^*} . \qquad (18)$$

where $W T^*$ and T^* are values from Tables I−IV.

Data, corresponding to this formula, are shown in the first rows of Tables V−VIII.

They provide lower bound estimation of the exact solution of Policy 2. The upper bound can be obtained from Tables I−IV. In addition, we have two cases when the length of the cycle in formula (3) can be calculated directly. For the minimal repair model ($q \quad 1$), we have $T_n \quad n \ 1/ \quad / \quad n$. If $q \quad 0$ (perfect repair), the length of the cycle is equal to n times of the Mean Time to Failure (MTTF) of the Weibull distribution $T_n \ \ n \quad 1\ 1/$.

Table V – Optimal maintenance cost (C_0 = 5, α = 2)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	4.023	3.9017	3.7533	3.600	3.249	2.793	2.096
Monte Carlo	4.128	3.992	3.846	3.690	3.3223	2.854	2.135
n	5	5	5	5	6	8	11
n^*	5	5	5	6	6	8	12
Length of cycle, T^*	2.18	2.255	2.340	2.439	3.010	4.204	7.025

Table VI – Optimal maintenance cost (C_0 = 3, α = 2)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	2.891	2.827	2.7440	2.667	2.473	2.208	1.774
Monte Carlo	3.010	2.935	2.854	2.767	2.563	2.275	1.817
n	2	3	3	3	4	5	8
n^*	3	3	3	4	4	5	9
Length of cycle, T^*	1.329	1.704	1.752	1.807	2.341	3.077	5.50

Table VII – Optimal maintenance cost (C_0 = 5, α = 5)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	5.010	4.957	4.7217	4.534	3.9443	3.2078	2.200
Monte Carlo	5.435	5.255	5.040	4.806	4.164	3.368	2.282
n	1	2	2	2	3	4	8
n^*	1	2	2	2	3	4	8
Length of cycle, T^*	0.920	1.142	1.190	1.248	1.681	2.375	5.258

Table VIII – Optimal maintenance cost (C_0 = 3, α = 5)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	3.112	3.114	3.045	3.082	2.7856	2.406	1.822
Monte Carlo	3.262	3.262	3.262	3.204	2.875	2.4692	1.857
n	1	1	1	2	2	3	6
n^*	1	1	1	1	2	3	6
Length of cycle, T^*	0.920	0.920	0.920	1.248	1.391	2.025	4.308

C. Maintenance Policy 3

The main result of our calculations is represented in Tables IX−XII for various values of replacement cost C_0 , Weibull shape parameter , and restoration factor q . All calculations are completed by the MC method using (4) (rows in the Tables with the name “Approximation”), and (1) (rows under the name “Monte Carlo”). We observed a relatively small difference between these data. The maximum of the difference is about 2.7%, and corresponds to a minimal repair model with $C_0 \ 3$ in our examples. The difference is decreasing when the restoration factor is decreasing, and does not depend on the shape parameter. In addition, in these tables, we provided the following calculation results: Monte Carlo Standard Error (SE), time interval T_3^* defined by Policy 3, and the expected length of replacement cycle T^* corresponding to the optimal maintenance cost.

Analyzing data from these tables, we can conclude that the minimal repair model is also not sensitive to restoration factor q. If the factor is changed from 1.0 to 0.9 (by 10%), the corresponding optimal cost decreases by 3% at most in our examples, when $C_0 \quad 5$, $\quad 2$.

Table IX – Optimal maintenance cost (C_0 = 5, α = 2)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	4.030	3.909	3.777	3.622	3.271	2.821	2.123
Monte Carlo	4.073	3.941	3.801	3.647	3.290	2.838	2.127
Monte Carlo SE	9.7E−3	9.2E−3	8.6E−3	8.0E−3	6.8E−3	5.4E−3	3.2E−3
T_3^*	1.914	2.021	2.198	2.304	2.765	3.368	7.090
Length of cycle, T^*	2.147	2.260	2.440	2.559	3.045	3.696	7.464

Table X – Optimal maintenance cost (C_0 = 3, α = 2)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	2.893	2.837	2.769	2.694	2.503	2.240	1.804
Monte Carlo	2.975	2.905	2.830	2.743	2.538	2.258	1.809
Monte Carlo SE	9.4E−3	9.0E−3	8.5E−3	7.9E−3	6.8E−3	5.3E−3	3.0E−3
T_3^*	1.347	1.383	1.560	1.631	1.914	2.694	4.892
Length of cycle, T^*	1.656	1.702	1.870	1.955	2.262	3.056	5.317

Table XI – Optimal maintenance cost (C_0 = 5, α = 5)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	5.215	5.102	4.935	4.709	4.113	3.340	2.276
Monte Carlo	5.285	5.146	4.969	4.736	4.126	3.347	2.277
Monte Carlo SE	9.1E−3	8.3E−3	7.5E−3	6.7E−3	5.3E−3	3.6E−3	1.9E−3
T_3^*	0.881	0.955	1.065	1.175	1.579	2.2403	4.995
Length of cycle, T^*	1.060	1.121	1.225	1.346	1.750	2.447	5.260

Table XII – Optimal maintenance cost (C_0 = 3, α = 5)

Restoration factor, q	1	0.9	0.8	0.7	0.5	0.3	0.1
Approximation	3.225	3.201	3.165	3.099	2.846	2.453	1.852
Monte Carlo	3.311	3.267	3.207	3.126	2.870	2.461	1.852
Monte Carlo SE	7.0E−3	6.6E−3	6.0E−3	5.4E−3	4.4E−3	3.2E−3	1.8E−3
T_3^*	0.734	0.808	0.845	0.918	1.175	1.726	3.820
Length of cycle, T^*	0.994	1.036	1.070	1.141	1.435	1.986	4.131

The difference declines if C_0 decreases.

Comparing calculation results for Policies 1 and 3, we conclude that Policy 3 is the most efficient in the case of the *minimal repair* model. The difference in the minimal cost changes from 9% to 17% in our examples. It is in-creasing if the periodical replacement cost is declining, and the Weibull shape parameter increases. However, in Policy 1, *replacement* is considered as *preventive* maintenance at the *scheduled* time, and can be less expensive than the *corrective* maintenance in Policy 3 at the time of failure. Our calculations show that, if the difference in replacement cost between the two considered cases is greater than 20%, Policy 1 is more efficient. In each particular practical case, an appropriate policy should be selected.

All results of the calculations show that Policy 3 is the most efficient among the considered three, as it is proven in [13] for the general case. However, the difference between Policies 3 and 2 is small. Its maximum was 2.8% for minimal repair when $C_0 \;\; 5$, $\;\; 5$.

Comparing calculation results for T_3^* corresponding to Policy 3 and optimal length of cycle T^* from Tables I–IV, we derived that the ratio almost does not depend on restoration factor q or shape parameter α. It can be represented as

$$T_3^* \quad kT^* , \tag{19}$$

where coefficient k depends on replacement cost C_0. If $C_0 \;\; 3$, then $k \;\; 0.8$; if $C_0 \;\; 5$, then $k \;\; 0.87$.

We approximated this logical dependence by a linear function $k \; 0.695 \; 0.035 C_0$, and tested it against the MC method according to data from Tables IX–XII. The difference in the calculation of minimal total cost was not greater than 0.35%. Therefore, (19) yields a good approximation for optimal time T_3^* of Policy 3.

In all the above calculation examples, corrective repair cost C_q and replacement cost C_0 did not depend on time or on the restoration factor. More general, practical cases will be considered in Section IV.

D. Procedure of finding optimal solution

Eventually, we implemented the described methodology in software which allows us to calculate optimal replacement times according to a maintenance policy selected by the user. At the first stage of the calculation, we use the approximate formulae corresponding to Policy 1. If Policy 2 or 3 is selected, then (18) and (19) are used in addition. This approximation was used as a first step. Next, a more accurate solution is obtained using the MC method. Only about 20 additional steps of the MC calculation are required to obtain an accurate optimal solution. We used multithreading in our software, which decreased the calculation time by about 4 times on our computer having 4 processors. It takes less than 1 second to calculate the final result. Because the optimization in question is done with respect to just one parameter, any optimization method can be used. We have used the method of gradient decent.

CONCLUSIONS

In this paper, we studied three of the most popular maintenance policies allowing us to define the optimal time to replace the unit with a new one. The failure process is described by the g−renewal Kijima model under the Weibull failure−time distribution function. The difficulty of the g–renewal process is that its g–renewal equation does not have a closed form solution in this case. We proposed two efficient solutions (an improved Monte Carlo method, and our previously obtained approximate solution), which enable an in–depth comparative analysis of the maintenance polices. The policies are compared for various values of the model parameters. The sensitivity of each model is studied with respect to the restoration factor. Practical cases are considered to show both the importance of maintenance optimization using a g−renewal model and the efficiency of the suggested Monte Carlo and approximate methods. The obtained calculation results can be used as a benchmark for developing other approximate methods.

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Анотація

ПОРІВНЯЛЬНИЙ АНАЛІЗ ОПТИМАЛЬНИХ ІНТЕРВАЛІВ ПЛАНОВО-ПОПЕРЕДЖУВАЛЬНИХ РЕМОНТІВ У РАМКАХ ПРОЦЕСУ G-ВІДНОВЛЕННЯ З БАЗОВИМ РОЗПОДІЛОМ ВЕЙБУЛЛА

Кривцов В. В., Євкін О. Ю.

Досліджуються два оптимальні методи планово-попереджувальних ремонтів у рамках g-відновлення з базовим розподілом Вейбулла. Перший базується на раніше отриманій наближеній формулі для g-відновлення, а другий є вдосконаленим методом Монте-Карло. Ці методи дозволяють поглибити порівняльний аналіз політик ремонту. Пропонується ефективний алгоритм для знаходження оптимального періоду для виконання планово-попереджувальних ремонтів.

Аннотация

СРАВНИТЕЛЬНЫЙ АНАЛИЗ ОПТИМАЛЬНЫХ ИНТЕРВАЛОВ ПЛАНОВО-ПРЕДУПРЕДИТЕЛЬНЫХ РЕМОНТОВ В РАМКАХ ПРОЦЕССА G-ВОССТАНОВЛЕНИЯ С БАЗОВЫМ РАСПРЕДЕЛЕНИЕМ ВЕЙБУЛЛА

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Исследуются два оптимальных метода планово-предупредительных ремонтов в рамках g-восстановления с базовым распределением Вейбулла. Первый основывается на ранее полученной формуле для g-восстановления, а второй является усовершенствованным методом Монте-Карло. Эти методы позволяют углубить сравнительный анализ политик ремонтов. Предложен эффективный алгоритм для нахождения оптимального периода для выполнения планово-предупредительных ремонтов.