MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

STATE BIOTECHNOLOGICAL UNIVERSITY

Faculty of Mechatronics and Engineering

Department of Physics and Mathematics

An Introduction to Discrete Mathematical Modelling

Guidelines for Students Studying a Course of Higher Mathematics in English

for the first (bachelor) level of Higher Education

Kharkiv
2024

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An Introduction to Discrete Mathematical Modelling: Guidelines for Students Studying a Course of Higher Mathematics in English / State Biotechnological University; authors: A.O. Pak, N.V. Smetankina, O.A. Mandrazhy. – Kharkiv, 2024. – 27 p.

Basic representations of Discrete Mathematical Modeling are given (several different types of models are available).

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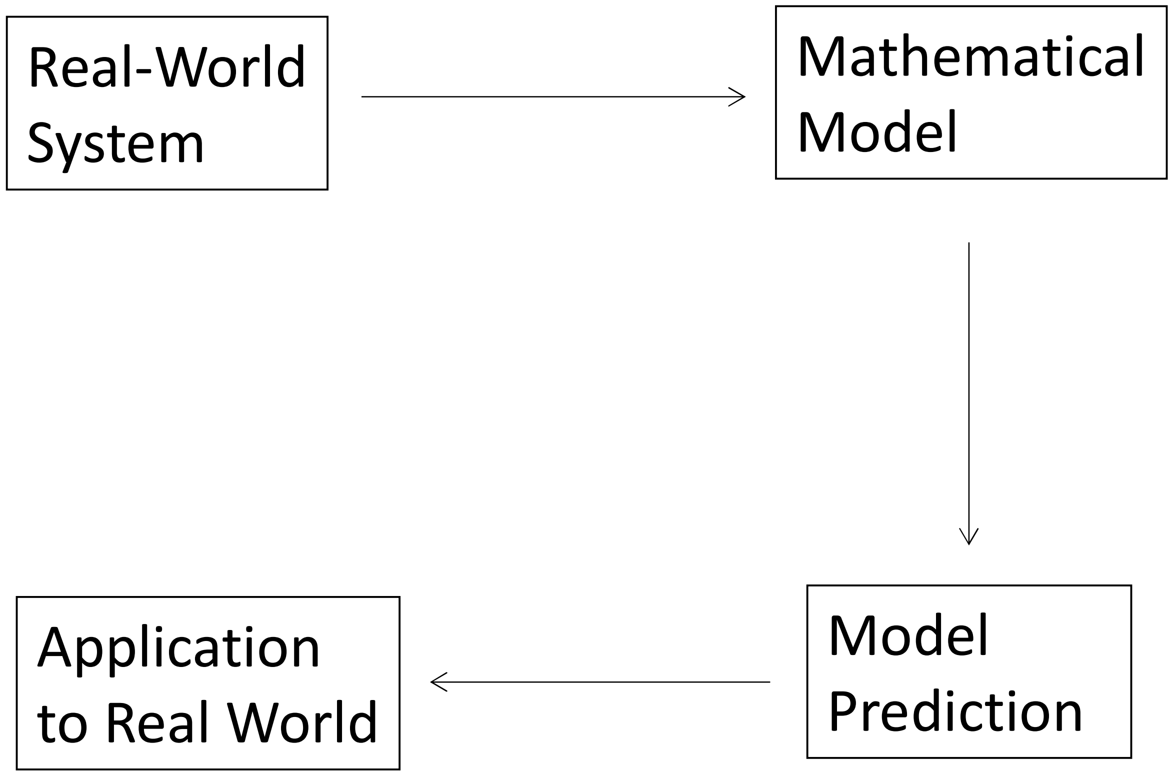
An Introduction to Discrete Mathematical Modelling

Use of Models



Models are often used to gain a better under-standing of real-world systems, often with a view to making decisions.

A model often contains *parameters* (values that can be measured) and *variables* (values can be chosen).

We introduce some different types of models.

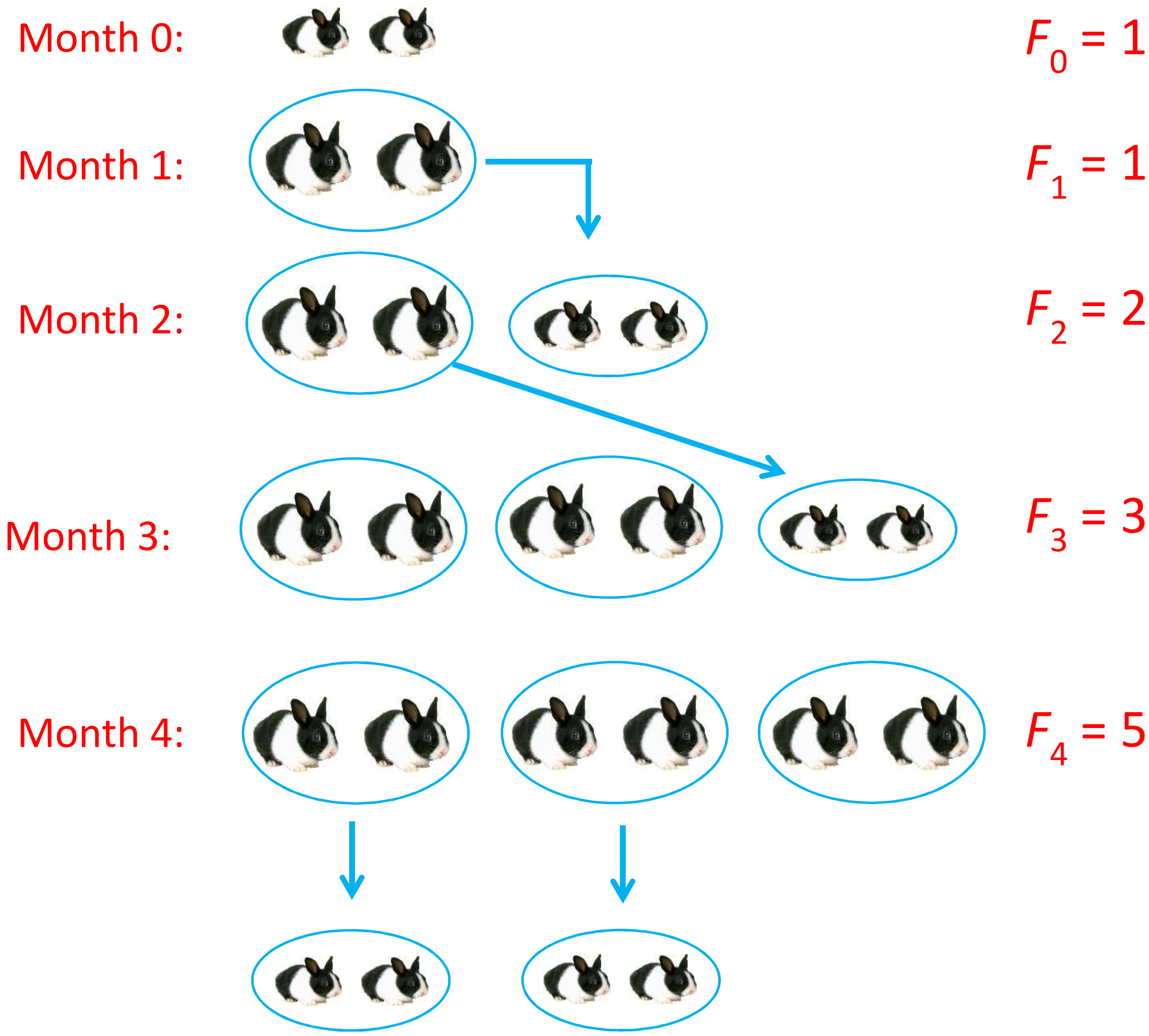
1. Models from Art/Nature

In his book, Leonardo Fibonacci (1175-1250) posed the following problem.

"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that each month each pair begets a new pair which from the second month on becomes productive?"

Let F_n denote the number of pairs of rabbits n months after the first pair was introduced.

By definition, $F_0 = 1$.



Continuing, we see that the number of pairs in month 5 is equal to the number of pairs in month 4, plus the number that reproduce, which is the number in month 3, i.e.,

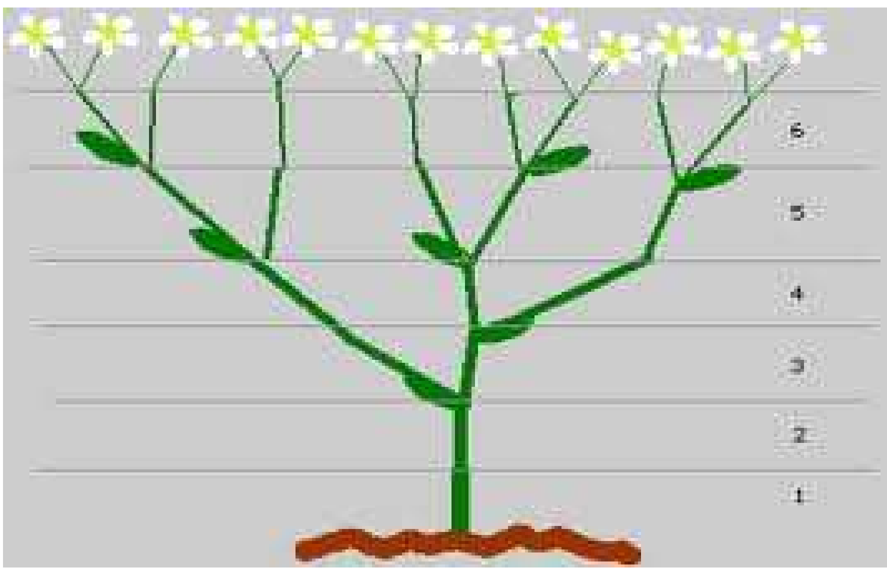
$F_5 = F_4 + F_3 = 5 + 3 = 8.$

The number of pairs of rabbits in successive months is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233,… , which are the Fibonacci numbers.

$$F_{n+2} = F_{n+1} + F_n \quad \text{where} \;\; F_0 = F_1 = 1.$$

Fibonacci’s rabbit problem is rather artificial, but there is a similar phenomenon in nature.

The *sneezewort* has to grow 2 months before it branches, and then it branches every month. The new shoot has to grow for 2 months before branching.



$$F_{n+2} = F_{n+1} + F_n \quad \text{where} \quad F_0 = F_1 = 1.$$

The above is a second-order difference equation. We try to solve it by setting $F_n = x^n$. Substituting

$$x^{n+2} = x^{n+1} + x^n$$

$$x^2 = x + 1$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

The general solution of the equation is

$$F_n = a\left[\frac{1+\sqrt{5}}{2}\right]^n + b\left[\frac{1-\sqrt{5}}{2}\right]^n$$

where a and b are arbitrary constants. To satisfy the initial conditions, we require

$$a + b = 1 \qquad a(1+\sqrt{5})/2 + b(1-\sqrt{5})/2 = 1$$

$$a + b = 1 \qquad a - b = 1/\sqrt{5}$$

$$a = (1 + \sqrt{5})/(2\sqrt{5}) \qquad b = -(1 - \sqrt{5})/(2\sqrt{5})$$

$$F_n = (1/\sqrt{5})[\,[(1+\sqrt{5})/2]^{n+1} - [(1-\sqrt{5})/2]^{n+1}]$$

$$F_n = (1/\sqrt{5})[\,[(1+\sqrt{5})/2]^{n+1} - [(1-\sqrt{5})/2]^{n+1}]$$

For large n, the second term in F_n is negligible,

$$F_n \approx (1/\sqrt{5})\,[(1+\sqrt{5})/2]^{n+1}$$

Let

$$\Phi = (1+\sqrt{5})/2 \approx 1.618034$$

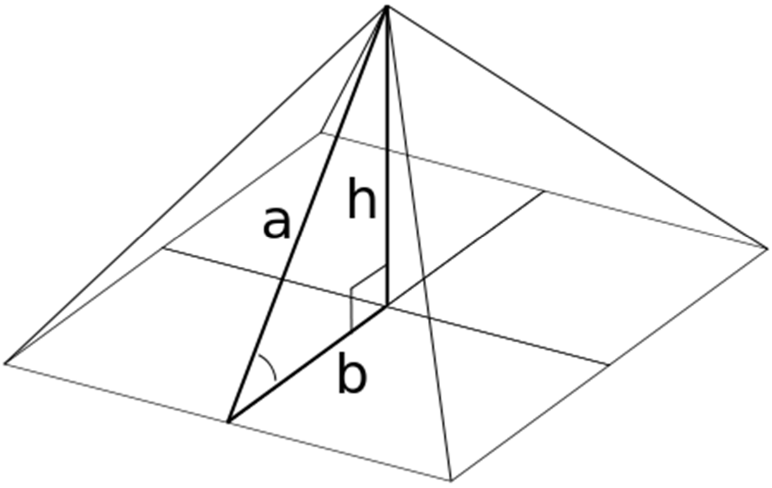
This value Φ is known as the ***Golden Ratio***. Note that

$$\frac{F_{n+1}}{F_n} \to \Phi \quad \text{as } n \to \infty.$$

Also

$$\Phi \approx \frac{1+\Phi}{\Phi}.$$

For the Great Pyramid of Khufu at Ghuza (2480 BC), it is said that "the square of the great pyramids height is equal to the area of its triangular lateral side".



Without loss of generality, let $b = 1$. Then

$$a^2 = b^2 + h^2 = 1 + h^2$$

by Pythagoras, and the assertion is that

$$h^2 = a.$$

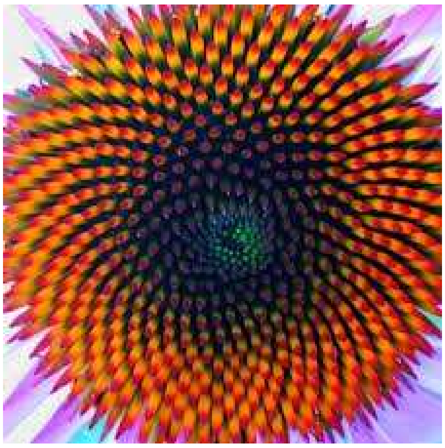
In this case,

$$a^2 = 1 + a$$

and therefore $a = \Phi$, and $h = \sqrt{\Phi}$.

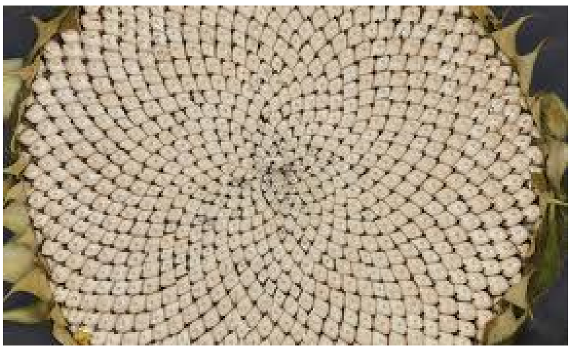
Actual sizes in feet are h = 481, b = 378, which gives a = 612. Since 612/378 = 1.62, this is a close approximation to Φ.

The pine cone exhibits 8 anti-clockwise spirals and 13 clockwise spirals. The parastichy number is (8,13).



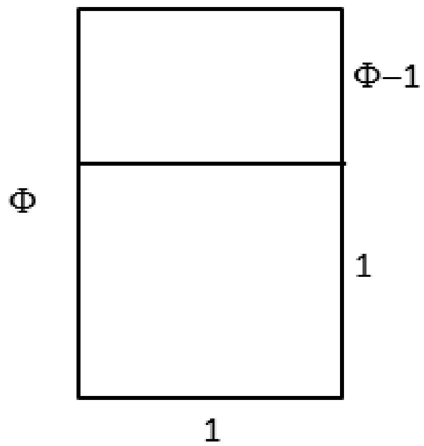
A coneflower (a type of daisy) seed head.

A sunflower seed head.



Larger sunflowers with parastichy numbers of (34, 55), (55, 89), (89, 144) and (144, 233) have been observed.

A *Golden Rectangle* is one in which the sides are in the proportion $1 : \Phi$.

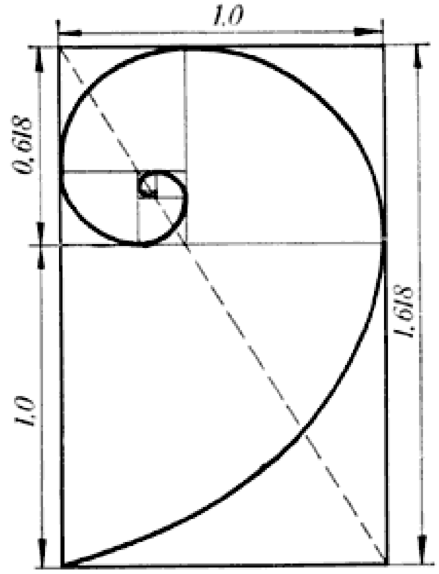


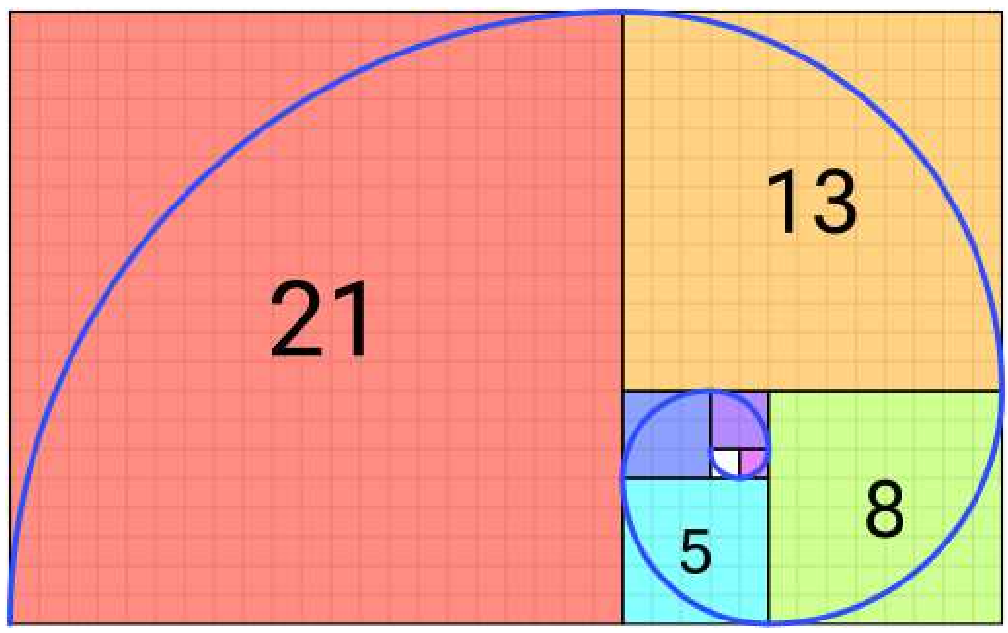
The smaller rectangle at the top has sides in the proportion $(\Phi - 1) : 1$.

However, note that

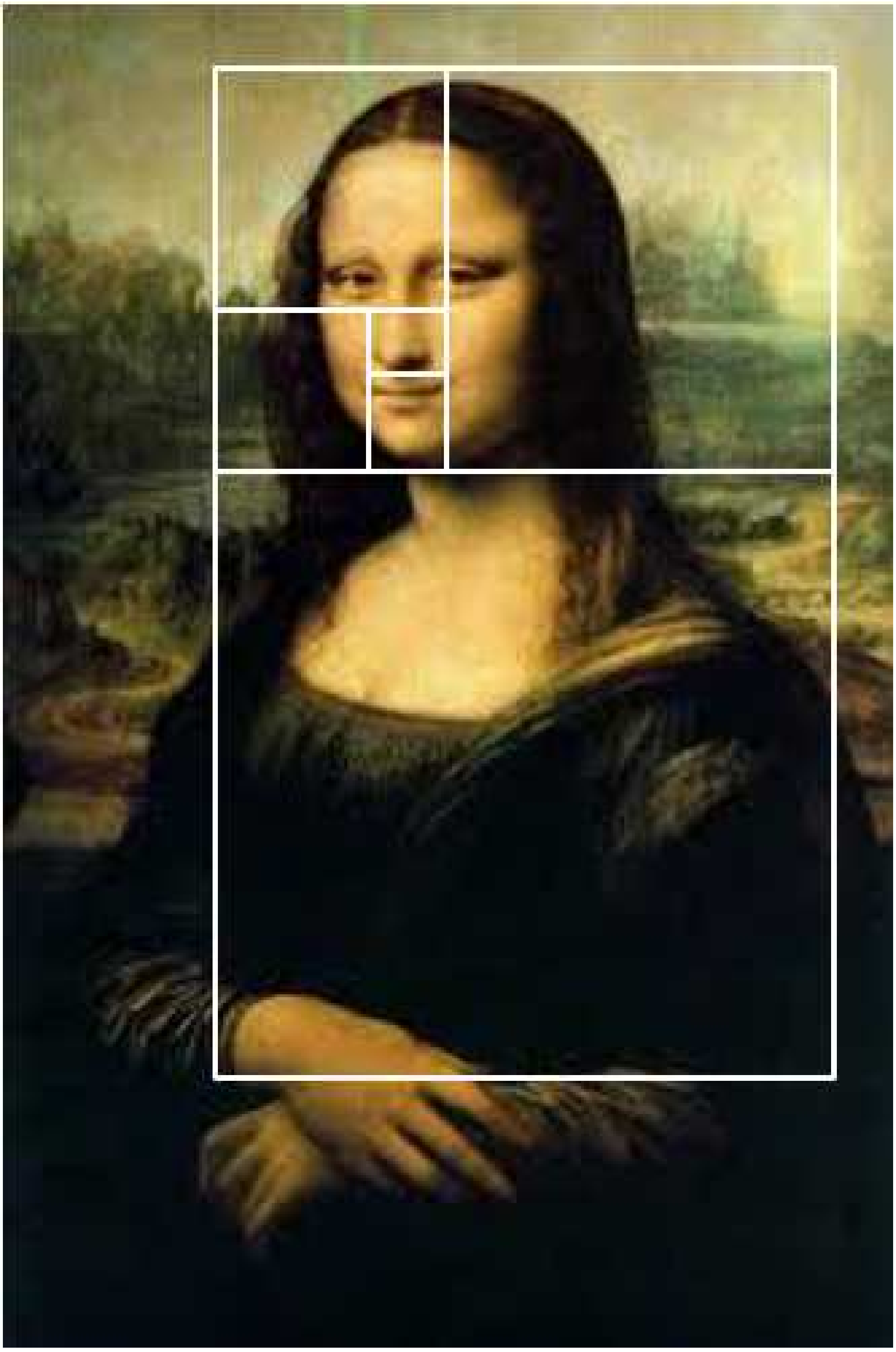
$$(\Phi - 1)/1 = 1/\Phi$$

Thus, the smaller rectangle is proportional to the original larger rectangle. The smaller rectangle can be divided in a similar way, and the newly formed rectangle will again have the same proportions are the original rectangle.





A Fibonacci spiral approximates the golden spiral using quarter-circle arcs inscribed in squares derived from the Fibonacci sequence [https://en.wikipedia.org/wiki/Golden_spiral].



Many well-known paintings, and the features within them, have proportions close to the Golden Rectangle. The width to the height of an adult face is an obvious example.

2. Queueing Models: One Step at a Time Changes

A queue is characterized by a person or object needing attention by a server. Queues occur in many practical contexts as well as queueing for service in shops. For example:

- products waiting for a machine in manu-facturing,
- aircraft waiting to take-off on a runway,
- patients waiting to see a doctor,
- customers waiting to speak to an agent at a call centre,
- vehicles waiting at a toll booth.

Queue to vote in a South African election.

A Basic Single Server Queue

[https://ru.dreamstime.com/photos-images/очередь.html]

Assumptions

- Arrivals occur at random at a rate λ, so that the probability of an arrival in an interval of length δt is $\lambda\delta t$.
- Service completions occur at random at a rate μ, so that the probability of a service completion in an interval of length δt is $\mu\delta t$.
- The probability of two or more events (arrivals or service completions) occurring in an interval of length δt is small (tends to zero when dividing by δt and letting $\delta t \to 0$). This term is written as $O(\delta t^2)$.
- For stability $\lambda < \mu$.

Let $p_n(t)$ denote the probability of n customers in the system at time t. For $n \geq 1$, we obtain

$$p_n(t+\delta t) = p_n(t)(1-\lambda\delta t)(1-\mu\delta t) + p_{n-1}(t)\lambda\delta t(1-\mu\delta t)$$
$$+ p_{n+1}(t)(1-\lambda\delta t)\mu\delta t + O(\delta t^2)$$

$$(p_n(t+\delta t) - p_n(t))/\delta t = -(\lambda + \mu)p_n(t) + \lambda p_{n-1}(t)$$
$$+ \mu p_{n+1}(t) + O(\delta t)$$

Consider the *steady state*, where probabilities do not depend on time, so $p_n(t) = p_n$. Let $\delta t \to 0$.

$$-(\lambda + \mu)p_n + \lambda p_{n-1} + \mu p_{n+1} = 0$$

Similarly, for $n = 0$, we obtain

$$p_0(t+\delta t) = p_0(t)(1-\lambda\delta t) + p_1(t)(1-\lambda\delta t)\mu\delta t + O(\delta t^2)$$

$$(p_0(t+\delta t) - p_0(t))/\delta t = -\lambda p_0(t) + \mu p_1(t) + O(\delta t)$$

Consider the steady state and let $\delta t \to 0$.

$$-\lambda p_0 + \mu p_1 = 0$$

$$-(\lambda + \mu)p_n + \lambda p_{n-1} + \mu p_{n+1} = 0 \qquad (1)$$

$$-\lambda p_0 + \mu p_1 = 0 \qquad (2)$$

Rewriting equation (1), and repeating this equation for decreasing indices *n*, we find that

$$\mu p_{n+1} - \lambda\, p_n = \mu p_n - \lambda p_{n-1}$$

$$\mu p_n - \lambda\, p_{n-1} = \mu p_{n-1} - \lambda p_{n-2}$$

etc., and finally

$$\mu p_2 - \lambda p_1 = \mu p_1 - \lambda p_0 = 0$$

where the final equation is obtained from (2).

We conclude that

$$p_n = (\lambda/\mu)p_{n-1} \qquad (3)$$

Applying (3) successively

$$p_n = (\lambda/\mu)p_{n-1} = (\lambda/\mu)^2 p_{n-2} = \ldots = (\lambda/\mu)^n p_0$$

Letting $\rho = \lambda/\mu$, where $\rho < 1$, which is known as the *traffic intensity*, our formula for p_n is

$$p_n = \rho^n p_0$$

$$p_n = \rho^n p_0$$

To obtain the actual probabilities (rather than have them expressed in terms of p_0), we use the fact that probabilities should sum to 1. Thus,

$$p_0 (1 + \rho + \rho^2 + \rho^3 + \dots) = 1$$

We now use the formula for the sum of an infinite geometric series:

$$p_0 = 1 - \rho$$

$$p_n = (1 - \rho)\rho^n$$

It is possible to derive formulas for different measures of performance such as the average waiting time of customers in the queue

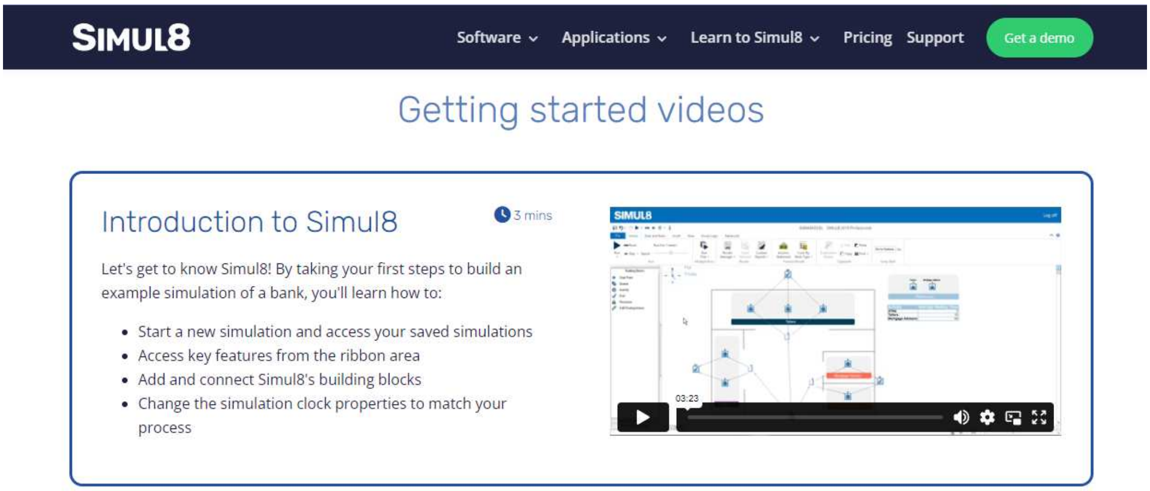
$$W_q = \lambda/[\mu(\mu - \lambda)]$$

Multiple-server queues have a single queue with the person at the front of the queue going to the service point when a server is available, as in a bank, rather than in supermarkets where each till has its own queue.

3. Simulation Models

A simulation model aims at replicating the behaviour of a real system, usually on a computer. Simulation models often involve a complex system of queues, that do not fall within the category of queueing models that can be studied using known analytical tools.

For example, Simul8 is the computer software that we use for simulation. See http://www.simul8.com/2min/



Practical examples of Simul8 are:

- allocating check-in desks at Glasgow airport using "hot-swapping",
- designing a bus station for Hertfordshire County Council,
- British Nuclear Group used simulation to find ways of moving nuclear waste to a safer location considering the trade off between time and cost,
- achieving better use of key resources at BUPA Parkway Hospital.

4. Optimization Models

In an optimization model, an *objective function* is to be maximized or minimized, subject to *constraints* on the values of the variables. The objective function and constraints contain both variables (values to be decided upon) and parameters (values given).

When the objective function and constraints are all linear, we have a *linear programming* problem, which is easy to solve. When some variables are also required to take integer values, then the resulting *integer (linear) programming* problem is hard to solve. Non-linear programming problems can also be hard depending on the nature of the non-linear functions.

As an example of an optimization model, we consider the problem of finding a ranking which best fits given data.

Example. Consider an arbitrary 6 Premier League teams. We aim to find a ranking of the teams that is most consistent between these teams.

Away Team

Home Team

	Man City	Man Utd	Arsenal	Spurs	N'castle	Chelsea
Man City	–	W	W	W	W	W
Man Utd	L	–	W	W	D	W
Arsenal	W	L	–	W	W	D
Spurs	L	L	W	–	W	D
N'castle	L	W	D	D	–	L
Chelsea	W	D	L	D	L	–

$$P = \begin{pmatrix} - & 6 & 3 & 6 & 6 & 3 \\ 0 & - & 6 & 6 & 1 & 4 \\ 3 & 0 & - & 3 & 4 & 4 \\ 0 & 0 & 3 & - & 4 & 2 \\ 0 & 4 & 1 & 1 & - & 3 \\ 3 & 1 & 1 & 2 & 3 & - \end{pmatrix}$$

The value of p_{ij} is the number of points scored by team *i* in the home and away matches against team *j*.

Note the data inconsistency: 2 better than 3,
3 better than 5,
5 better than 2.

Let *n* denote the number of teams, and

$$x_{ij} = \begin{cases} 1 & \text{if team } i \text{ is ranked above team } j, \\ 0 & \text{otherwise.} \end{cases}$$

The optimization model is

$$\text{Maximize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_{ij}$$

$$\text{subject to} \quad x_{ij} + x_{ji} = 1 \qquad \text{for all } i, j$$

$$x_{ij} + x_{jk} + x_{ki} \le 2 \qquad \text{for all } i, j, k$$

$$x_{ij} \in \{0, 1\} \qquad \text{for all } i, j$$

$$x_{ii} = 0 \qquad\qquad\qquad \text{for all } i$$

A solution of the model is the ranking (1, 2, 3, 4, 5, 6), which is slightly different from the points ranking among these 6 teams.

This type of model can be applied to input/ output matrices of industrial sectors to find the most important ones, and to classify archaeological finds according to their age.

Notes

Educational Edition

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