УДК 621.316.3

ON BAYESIAN ESTIMATION OF RELIABILITY FUNCTION FOR LIFETIME DISTRIBUTIONS

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This paper is a sequel to [1], wherein we proposed a simple procedure to construct the joint prior and posterior distributions of Weibull parameters based on the underlying reliability function estimates in two time cross–sections. In this paper, we extend the procedure in three aspects: a) the prior data can now be taken in terms of a simple probability paper plot, b) the posterior now includes not only posterior estimates of distribution parameters, but also the posterior estimate of the underlying reliability function along with the respective credibility intervals, and c) we show that the proposed procedures can be applied to any parametric lifetime distribution, not necessarily limited to the location–scale family.

Introduction. Bayesian estimation of parametric reliability functions has been considered by many authors. A critical first step is the specification of the prior distribution of the reliability function parameter(s). Some authors omit this step and treat it as a given, i.e., the (joint) prior distribution of parameter(s) is assumed to be known. A few authors do discuss procedures of constructing the (joint) prior distribution based on expert estimates or life test data. Under the assumption of independence, one can specify marginal priors of individual distribution parameters, thus arriving to their joint prior. For example, Canavos & Tsokos [2] assume independent priors for the shape (uniform) and scale (inverse gamma) parameters of the Weibull distribution. Meeker & Escobar [3] discuss constructing the joint prior distribution of the Weibull distribution based on the Weibull shape parameter and a certain quantile of the underlying CDF. They make a point that in case of heavy censoring (which is often the case in reliability data analysis), specifying the shape parameter and an appropriately chosen quantile of the underlying distribution would be more robust than specifying individual priors for the shape and the scale parameters.

Kaminskiy & Krivtsov [1] proposed a simple procedure to construct the joint prior distribution of Weibull parameters based on the estimates of the underlying reliability function in two time cross–sections. The outcome of the procedure was the joint prior distribution of Weibull parameters as well as the posterior point estimate of the underlying reliability function at any time of interest. The procedure turned out to be quite useful in practical applications. As of 2019, it has been cited in over 90 publications [4] and was implemented in the 2013 release of the JMP® reliability analysis software package by the SAS Institute [5]. It must be mentioned that the procedure discussed in [3] is also implemented in the same package as an alternative method of specifying the prior.

In this paper [6], we extend the procedure proposed in [1] in three aspects: a) the input (prior) data can be taken in the form of a simple probability paper plot, b) the output includes not only posterior estimates of the distribution parameters, but also the posterior estimate of the underlying reliability function along with the respective credibility intervals, and c) we show that the proposed procedures can be applied to any parametric lifetime distribution. These extensions make the procedure much more usable for practical engineering applications.

Proposed approach. Without loss of generality, consider a two–parameter lifetime distribution with the cumulative distribution function $F(t;\ \alpha,\ \beta)$, wherein α and β are the scale and the shape parameters, respectively.

A penultimate step in the Bayesian estimation of the reliability function is the posterior joint density of the aforementioned parameters, which is given by:

$$\pi(\alpha,\beta) = \frac{\pi_0(\alpha,\beta)\, L(data \mid \alpha,\beta)}{\iint \pi_0(\alpha,\beta)\, L(data \mid \alpha,\beta)\, d\alpha\, d\beta}, \qquad (1)$$

where $\pi(\alpha,\ \beta)$ and $\pi_0(\alpha,\ \beta)$ – the posterior and prior joint PDF of the parameters distribution, respectively;

L – the likelihood function relative to current data.

Since the denominator in (1) does not depend on parameters α and β (as it integrates to a constant), it is sufficient to estimate only the numerator. Assume that the *prior data* about parameters α and β is available in a form of a random sample from the underlying lifetime distribution:

$$(t_1, \delta_1), (t_2, \delta_2), (t_3, \delta_3), \dots, (t_n, \delta_n), \qquad (2)$$

where t_i – observed lifetime;

δ_i – censoring indicator, such that $\delta_i = 1$ if t_i is a complete observation or $\delta_i = 0$ if t_i is a right–censored observation.

Further assume that *current data* about parameters α and β are available in a form of another random sample:

$$(T_1, \gamma_1), (T_2, \gamma_2), (T_3, \gamma_3), \dots, (T_m, \gamma_m). \qquad (3)$$

where T_i – observed lifetime;

γ_i – censoring indicator.

The following bootstrap simulation procedure is proposed to estimate $\pi_0(\alpha,\beta)\, L(data \mid \alpha,\beta)$ as well as the posterior reliability function.

Random resampling with replacement from sample (3) is performed n times. A sample is generated at each step i as:

$$(t_{1_i}, \delta_{1_i}), (t_{2_i}, \delta_{2_i}), (t_{3_i}, \delta_{3_i}), \dots, (t_{n_i}, \delta_{n_i}). \qquad (4)$$

Distribution parameters are estimated using the maximum likelihood procedure:

$$(\hat{a}_i, \hat{\beta}_i) = \underset{a_i, \beta_i}{\operatorname{argmax}} Ln\left(L\left((t_{1_i}, \delta_{1_i}), (t_{2_i}, \delta_{2_i}), \dots, (t_{n_i}, \delta_{n_i})\right)\right), \quad (5)$$

where the likelihood function L is defined as:

$$L = \prod_j f(t_j, a, \beta)^{\delta_j} (1 - F(t_j, a, \beta))^{1-\delta_j}, \quad (6)$$

Then, the posterior likelihood function can be calculated as:

$$LP_i = L(\hat{a}_i, \hat{\beta}_i, (T_1, \gamma_1), (T_2, \gamma_2), (T_3, \gamma_3), \dots, (T_m, \gamma_m). \quad (7)$$

The sample reliability function is:

$$S_i = 1 - F(t; \widehat{\alpha_1}, \widehat{\beta_1}). \quad (8)$$

The posterior joint distribution of parameters is estimated as:

$$\pi_0(a, \beta) L(data | a, \beta) = $$
$$= \sum_i 1_{\hat{a}_i \le a, \hat{\beta}_i \le \beta}(\hat{a}_i, \hat{\beta}_i) LP_i / \sum_i LP_i. \quad (9)$$

The posterior reliability function $R(S,t)$ is obtained as:

$$R(S, t) = \sum_i 1_{S_i \le S}(S_i)/n \quad (10)$$

The lower LCL_S and upper UCL_S two–sided 100% (1–α) credible limits are found as those satisfying the following equations:

$$R(LCL_S, t) = \alpha/2 \quad (11)$$
$$R(UCL_S, t) = 1 - \alpha/2. \quad (12)$$

Numerical Example. In this section, the Weibull distribution is used as an example, but the procedure being proposed can be extended to any parametric lifetime distribution (not limited to the location–scale family). In Figure 1, consider a Weibull probability plot of a random sample as in (2) with 39 complete and 11 censored failure times. The MLE estimates of the shape and scale parameters of this sample are $\hat{\beta} = 1.7$, $\hat{\alpha} = 43.8$, respectively.

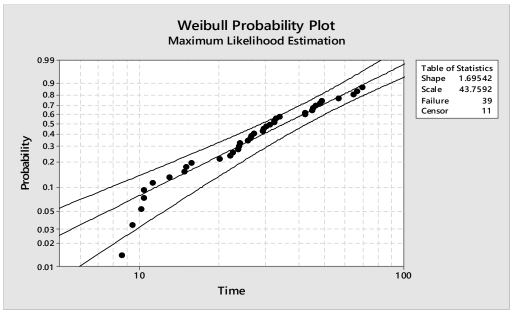

Figure 1 – A random Weibull sample to be used for constructing the joint prior distribution of Weibull parameters

As discussed in Section II, we obtain bootstrap samples (4) of the same size as the original sample – see Figure 2. To each bootstrap sample, we fit a Weibull distribution, thus obtaining a pair of Weibull parameters (5). By repeating the sampling procedure n times, we obtain n pairs of Weibull parameters, on which their joint distribution is constructed.

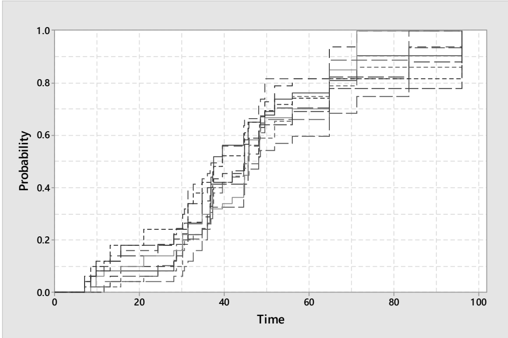

Figure 2 – Some bootstrap samples from the Weibull sample in Fig. 1

Figure 3 shows the joint prior distribution of Weibull parameters constructed based on n = 10,000 bootstrap samples from the sample in Figure 1. The highest prior density corresponds to α = 50.9, and β = 2.16.

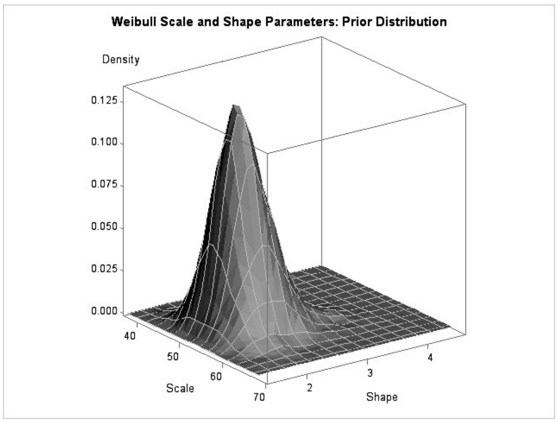

Figure 3 – The joint prior distribution of Weibull parameters constructed based on 10,000 bootstrap simulations from the Weibull sample in Fig. 1.

Using the following Weibull sample, see Figure 4, as the current data (3) we obtain the joint posterior distribution of Weibull parameters based on (9) – see Figure 5. The highest posterior density now corresponds to α = 52.1, and β = 2.05.

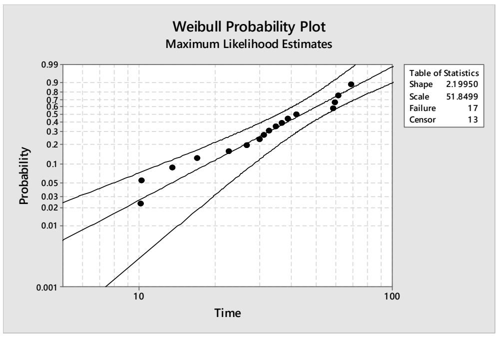

Figure 4 – A random Weibull sample to be used as current data for constructing the joint posterior distribution of Weibull parameters

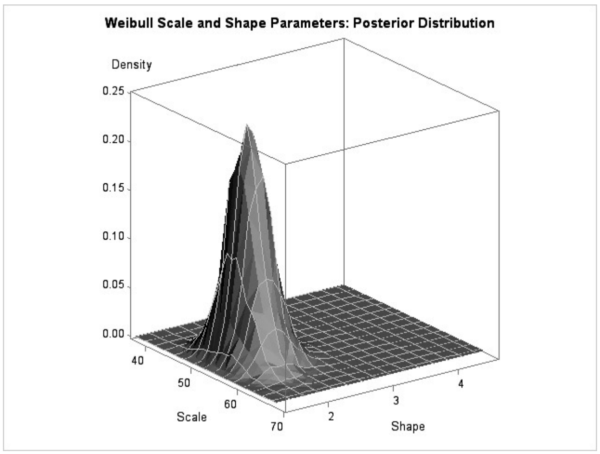

Figure 5 – The joint posterior distribution of Weibull parameters

Finally, using (10)–(12), we obtain the posterior Weibull reliability function with respective credibility intervals – see Figure 6.

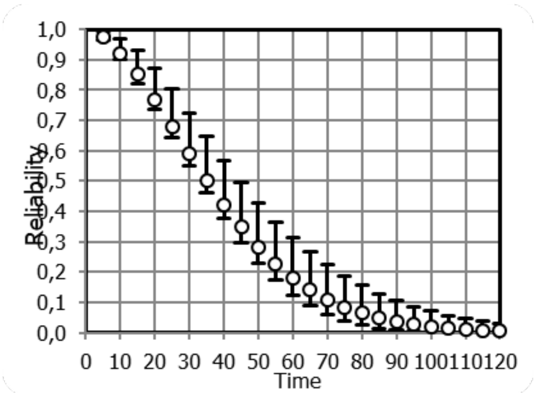

Figure 6 – The posterior Weibull reliability functions with 95% two–sided credibility intervals

Concluding Remarks. In practical reliability engineering, construction of the joint prior PDF for the parameters of the underlying lifetime distribution is a challenging task. This paper presented a simple procedure of Bayesian estimation of the Weibull distribution based on a single random sample characterizing prior data and a single random sample characterizing current data.

The Weibull distribution is discussed as an illustrative example, but the proposed procedure can be extended to any parametric lifetime distribution, not necessarily limited to the location–scale family. Moreover, it can even be used with empirical distributions without assuming any parametric structure.

Acknowledgement. We would like to acknowledge the review and valuable suggestions of Dr. Mark Kaminskiy of NASA Goddard Spaceflight Center, Greenbelt, MD, USA.

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Анотація

БАЙЕСІВСЬКА ОЦІНКА ФУНКЦІЇ НАДІЙНОСТІ ДЛЯ ТИПУ ТРИВАЛОСТІ ЖИТТЯ

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Ця стаття - продовження нашої попередньої роботи, в якій ми запропонували Баєсовську процедуру побудови спільного апріорного і апостеріорного розподілу параметрів Вейбулла-Гнеденко, покладаючись на оцінки функції надійності в 2-х перерізах вісі часу. У цій роботі ми розширили процедуру в трьох аспектах: а) апріорні дані можуть тепер бути представлені у вигляді "сирих" напрацювань до відмови, б) апостеріорні оцінки тепер включають не лише параметри розподілу, але і саму апостеріорну функцію надійності разом з відповідними інтервалами достовірності, в) ми показуємо, що запропонована процедура може бути застосована не лише до розподілу Вейбулла-Гнеденко, але і до усього сімейства масштаб-форма.

Аннотация

БАЙЕСОВСКАЯ ОЦЕНКА ФУНКЦИИ НАДЕЖНОСТИ ДЛЯ РАСПРЕДЕЛЕНИЙ ТИПА ВРЕМЕНИ ЖИЗНИ

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Данная статья - продолжение нашей предыдущей работы, в которой мы предложили Байесовскую процедуру построения совместного априорного и апостериорного распределения параметров Вейбулла-Гнеденко, полагаясь на оценки функции надежности в 2-х временных сечениях. В этой работе мы расширили процедуру в трех аспектах: а) априорные данные могут теперь быть представлены в виде "сырых" наработок до отказа, б) апостериорные оценки теперь включают не только параметры распределения, но и саму апостериорную функцию надежности вместе с соответствующими интервалами достоверности в) показано, что предложенная процедура может быть применена не только к распределению Вейбулла-Гнеденко, но и ко всему семейству масштаб-форма.