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APPROXIMATE SOLUTION TO G – RENEWAL EQUATION WITH UNDERLYING WEIBULL DISTRIBUTION

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An important characteristic of the g–renewal process, and of great practical interest, is the g–renewal equation, which represents the expected cumulative number of recurrent events as a function of time. The problem is that the g–renewal equation does not have a closed form solution, unless the underlying event times are exponentially distributed. The Monte Carlo solution [10], although exhaustive, is computationally demanding. This paper offers a simple-to-implement (in an Excel spreadsheet) approximate solution, when the underlying failure–time distribution is Weibull. The accuracy of the proposed solution is in the neighborhood of 2%, when compared to the respective Monte Carlo solution. Based on the proposed solution, we also consider an estimation procedure of the g–renewal process parameters.

ACRONYMS

CDF	cumulative distribution function;
CIF	cumulative intensity function;
ECIF	empirical cumulative intensity function;
GPR	generalized renewal process;
HPP	homogeneous Poison process;
IFR	increasing failure rate;
MC	Monte Carlo;
NHPP	non-homogeneous Poison process;
ORP	ordinary renewal process;
PDF	probability density function;
SE	standard error;
SS	sum of squares.

NOTATION

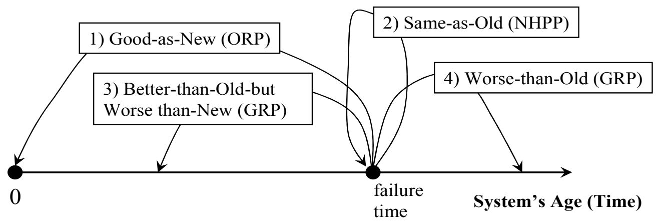
V_n, S_n – age of the system after and before the n–th repair, respectively;
q – restoration (or repair effectiveness) factor;
t – time;
$W(t)$ – g–renewal function denoting the expected cumulative number of events (failures);
$f(t)$ – probability density function;
$F(t)$ – cumulative distribution function;
λ, α – respectively, the scale and the shape parameters of Weibull distribution;
$\Gamma(\xi)$ – Gamma function;
μ, σ – the mean and the standard deviation of the failure time distribution;
N – number of simulations;
$P_L(t)$, $Q_M(t)$ – polynomials with order L and M, respectively;
$a, b, c, \gamma, A, B, C, D$ – numerical constants.

INTRODUCTION. In repairable system reliability analysis, one could consider four states, to which a system can be repaired upon a failure. These are: 1) "good-as-new" 2) "same-as-old", 3) "better-than-old-but-worse-than-new", 4) "worse-than-old". If upon a failure, a repairable system is restored to as "good-as-new" condition and the time between system failures can be treated as an independent and identically distributed (IID) random variable, then the failure occurrence can be modeled by the *Ordinary Renewal Process* (ORP). If upon a failure the system is restored to the "same-as-old" condition, then the appropriate model to describe the failure occurrence can be the *Non-Homogeneous Poisson Process* (NHPP). The time between consecutive failures, in this case, is not an IID random variable. A more general model is the so-called *Generalized Renewal Process* (GRP), which treats ORP and NHPP as special cases – see Figure 1.

The GRP or *g–renewal process*, originally introduced by Kijima and Sumita [13,14], has gained an increasing popularity in modeling and analysis of recurrent events, specifically in reliability applications [5, 8, 11, 20]. The GRP is introduced using the notion of *virtual age*:

$$V_n = qS_n ,$$

where V_n and S_n is the system's age after and before the n-th repair, respectively, and q is the *restoration* (or *repair effectiveness*) *factor*.



Figur 1 - Repair Assumptions & Respective Point Process Models

It is clear that for $q = 0$, the age of the system after the repair is "re-set" to zero, which corresponds to the ORP. With $q = 1$, the system is restored to the "same-as-old" condition, which is the NHPP. The case of $0 < q < 1$ corresponds to the intermediate "better-than-old-but-worse-than-new" repair assumption.Finally, with $q > 1$, the virtual age is $A_n > S_n$, so that the repair damages (ages) the system to a higher degree than it was just before the respective failure, which corresponds to the "worse-than-old" repair assumption. As such, all four considered cases of q can be modeled by the GRP.

Under the GRP, the expected number of events (failures) in (0, t] is given by a solution of the so-called g–renewal function [14]:

$$W(t)=\int_0^t\left(g(\tau\,|\,0)+\int_0^\tau w(x)g(\tau-x\,|\,x)dx\right)d\tau,$$

where

$$g(t \mid x) = \frac{f(t+qx)}{1-F(qx)}, \quad t, x \ge 0; \quad w(x) = \frac{dW(x)}{dx};$$

and $F(t)$ and $f(t)$ are the *cumulative distribution function* (CDF) and *probability density function* (PDF) of the underlying failure time distribution. Note that $g(t|0) = f(t)$.

The closed form solution of the g–renewal equation does not exist, and even numerical solutions are difficult to obtain, since each equation contains a recurrent infinite system [6]. The Monte Carlo (MC) solution discussed by Kaminskiy and Krivtsov [10], although exhaustive, is computationally demanding. The present paper offers a much simpler approximate solution, which can be implemented in an Excel spreadsheet. Its accuracy approaches the MC solution for all practical purposes.

TWO-POINT PADE APPROXIMANTS FOR ORDINARY RENEWAL EQUATION. In the first step, we consider ordinary renewal process, which is used to model the situation with restoration to "good-as-new" condition (the so-called *perfect repair* assumption). This process corresponds to a particular case of the g–renewal process, when restoration factor, $q = 0$. The time-depended renewal function, $W_0(t)$, gives the expected numbers of replacements and satisfies the integral equation

$$W_0(t) = F(t) + \int_0^t F(t-\tau)\, dW_0(\tau) \quad (1)$$

where $F(t)$ is the CDF of the underlying failure time distribution. In this paper, we consider the most popular Weibull distribution with the CDF expressed by

$$F(t) = 1 - e^{-(\lambda t)^{\alpha}} \quad (2)$$

in the time interval $t \ge 0$. The scale and shape parameters are restricted to the range $\lambda > 0$ and $\alpha > 0$, respectively.

A solution of Equation (1) was obtained in [18] as a series expansion

$$W_0(t) = \sum_{k=1}^{\infty} a_k (\lambda t)^{k\alpha}, \quad (3)$$

Coefficients of the series are defined by a simple recursive procedure

$$\begin{aligned} A_1 &= \gamma_1 \\ A_2 &= \gamma_2 - \gamma_1 A_1 \\ &\dots \\ A_k &= \gamma_k - \sum_{i=1}^{k-1} \gamma_i A_{k-i} \end{aligned} \quad (4)$$

where $\gamma_k = \frac{\Gamma(k\alpha+1)}{k!}$, $\Gamma(x)$ is Gamma function, and

$$a_k = \frac{(-1)^{k-1} A_k}{\Gamma(k\alpha+1)}$$

The partial sum of the series (3) with several terms

$$W_0(t,\alpha) \approx a_1(\lambda t)^{\alpha} + \dots + a_n(\lambda t)^{n\alpha} \quad (5)$$

gives a good approximation of the solution for small values of time and can be considered an asymptotic representation of the solution when $\lambda t \to 0$. If $\lambda t > 1$, the convergence of the series is very slow (especially if $\alpha > 1$) and additional enhancement of the solution is required. For this reason, many authors considered another well known asymptotic representation [2] for large values of t:

$$W_0(t,\alpha) \approx \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} \quad \text{if } t \to \infty \quad (6)$$

where μ, σ are the mean and the standard deviation of the underlying failure time distribution, respectively. For the Weibull distribution, we have:

$$\mu = \frac{1}{\lambda}\Gamma(1+\frac{1}{\alpha}), \quad \sigma^2 = \frac{1}{\lambda^2}\Gamma(1+\frac{2}{\alpha}) - \mu^2.$$

The asymptotic expansions (5) and (6) complement each other, but they are not accurate enough for the entire range of t from zero to infinity. For that reason, solution (5) was represented as a modified Padé function in [7]. The [L/M] Padé function [1] is defined as a rational function, where numerator $P_L(t)$ and denominator $Q_M(t)$ are polynomials with order L and M, respectively. The Padé functions were used in [4] to obtain a better solution in time interval $[0, t_0]$ for the renewal process with truncated Gaussian underlying distribution function. Value t_0 was defined as a switch-over point from Padé function to asymptotic function (6) under the condition that the obtained spline is close to the exact solution. This method was extended in [7] to the case when the underlying distribution function is Weibull. In addition, modified Padé functions [7] are used to construct the uniform interpolant joining (5) and the first term of asymptotic (6). The absolute relative error in the latter case reaches 5%, if $\alpha = 5$ and increases with the increase of shape parameter α. In both cases, some additional significant calculations are required to obtain coefficients of Padé function.

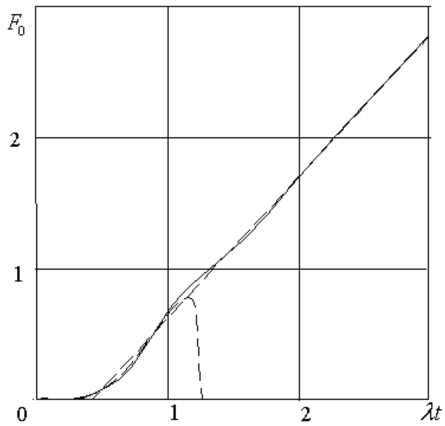
We suggest using two-point Padé functions in the form

$$W_0(t) \approx \frac{a_1(\lambda t)^{\alpha} + \dots + a_n(\lambda t)^{n\alpha} + A(\lambda t)^{n\alpha} t}{1 + B(\lambda t)^{n\alpha}} \quad (7)$$

This simple function has the same asymptotic expansion as (5) when $t \to 0$. To satisfy asymptotic representation (6) we have to put

$$B = \frac{2a_n\mu^2}{\sigma^2 - \mu^2}, \; A = \frac{B}{\mu}$$

To make sure that the rational function (7) does not have poles, we selected number n such that $a_n > 0$ if $\sigma^2 - \mu^2 > 0$ and $a_n < 0$ if $\sigma^2 - \mu^2 < 0$. Under this condition, $B > 0$ and the denominator of the rational function is not equal to 0 for any values of time. The accuracy of (7) depends on n. We defined the optimal value $n = 17$ if $\alpha > 1$ and $n = 8$ if $\alpha \le 1$. In Figure 2, the result of the calculation is represented for the case when $\alpha = 5$. The solid curve corresponds to the Monte Carlo solution, while the dashed curves represent asymptotic formulas (5) and (6).



Figur 2 - CIF of ordinary renewal process ($\alpha = 5$).

The accuracy of approximation (7) with respect to the "exact" Monte Carlo solution is shown in Table 1. The results of the proposed two-point Pade approximation (7) and the respective Monte Carlo method (with number of trials $N = 10^7$) is represented at corresponding time values (under $\lambda = 1$) when the maximum error was reached. The last row of the table shows the standard error (SE) of the Monte Carlo solution.

Table 1. The renewal function using two-point Pade approximation (7) compared to Monte Carlo solution under various values of α and with λ =1

Shape parameter, α	3	4	5	6	7
Two-point Pade	1,3453	1,0703	1,2630	1,2396	0,7907
Monte Carlo	1,3597	1,1087	1,2136	1,1590	0,9755
Maximum error	1%	3,5%	4%	7%	19%
Time at max error	1,6	1,4	1,6	1,6	1,2
Monte Carlo SE	0,00074	0,00056	0,00062	0,00058	0,00045

Approximation (7) is accurate enough for most practical cases, if shape parameter $\alpha \le 5$. For greater values of α , the renewal function is oscillating [3] and the simple formula cannot be derived in a general case. More accurate (but also more complicated) methods can be applied, e.g., [7], [3]. A simple approximation for the renewal function with the underlying increasing failure rate (IFR) distribution ($\alpha > 1$) was obtained in [9] for a reasonably practical time interval.

We improved (7) in interval $3 \le \alpha \le 5$ by adding an additional term C to the numerator of (7):

$$W_0(t) \approx \frac{a_1(\lambda t)^{\alpha} + \ldots + a_n(\lambda t)^{n\alpha} + A(\lambda t)^{n\alpha} t + C}{1 + B(\lambda t)^{n\alpha}} \tag{8}$$

where

$$C = \begin{cases} (\alpha - 3)\dfrac{(1,428 - (\lambda t))(\lambda t)^{n\alpha - 5}}{1 + (\lambda t)^{\alpha}} & if\ \alpha > 3 \\ 0 & if\ \alpha \le 3 \end{cases}$$

All coefficients in this formula were selected to minimize the maximum relative error with respect to the Monte Carlo solution. The maximum error decreases to 2,7% in interval $3 \le \alpha \le 5$ after this adjustment.

APPROXIMATE CUMULATIVE INTENSITY FUNCTION OF G – RENEWAL PROCESS. In general, the problem of approximation of the cumulative intensity function (CIF) of the g-renewal process is much more complicated. Because of the additional restoration factor q we have to consider a function of 3 variables $F(t, \alpha, q)$. The scale parameter can be eliminated by appropriate substitution of the time variable. Vaurio obtained an approximate solution of the g-renewal equation in [19]. However, our Monte Carlo simulations revealed a considerable limitation of his solution: it is accurate, if virtual time *qt* is small or large, but it yields a significant error when $qt \sim 1$. Our approach is based on the following properties of the CIF.

3.1. Restoration factor q

We assume to have an already reasonably accurate solution $W_0(t, \alpha)$, if *q=0*, that is the case of the ORP. If $q = 1$, we have the Nonhomogeneous Poisson Process the with exact solution

$$W_1(t, \alpha) = t^{\alpha} \tag{9}$$

Our attempt is to construct an approximate solution in interval $0 \le q \le 1$. We suggest an heuristic formula for the CIF approximation:

$$W(t, \alpha, q) \approx W_0(t, \alpha) + q^{\gamma} \left(W_1(t, \alpha) - W_0(t, \alpha) \right) \tag{10}$$

For any $\gamma \neq 0$ we have included two previous cases when $q = 0$ and *q=1*. To construct function *γ* we first consider general properties of the CIF.

3.2. *Time variable t*

The CIF function has an asymptotic solution

$$W(t, \alpha, q) \approx t^{\alpha} \ \ if\ t \to 0 \tag{11}$$

for any α and *q*. Formula (10) satisfies this asymptotic because $W_1(t, \alpha) \approx W_0(t, \alpha)$ if $t \to 0$.

3.3. *Shape parameter* α

Formula (10) is the exact solution for $\alpha = 1$ with any values of *q* and *t*, because it corresponds to the exponential distribution function and $W_1(t) = W_0(t)$ in this case. In addition:

$$W(t) \to W_0(t) \ \ if\ \alpha \to 0 \tag{12}$$

for any values of *q* and *t*. To satisfy this last property we have to put $\gamma \to 0$ if $\alpha \to 0$ **.**

3.4 *Defining function γ*

To interpolate the CIF, we use Padé approximant for *γ* in the following form

$$\gamma = \frac{a\alpha + b\alpha^2}{1 + c\alpha} \tag{13}$$

assuming that *γ* depends only on shape parameter α . We will find limitations of this assumption later on.

Property (12) is included in (13). We can define the coefficients of rational function (13) if solution is known at some point. For each *i*-th point, we can use (10). Solving it for *γ*, we obtain the following:

$$\gamma_i = \frac{\ln\left((W(t_i) - W_0(t_i)) / \left(W_1(t_i) - W_0(t_i) \right) \right)}{\ln q_i} \tag{14}$$

and, finally, for the coefficients of Padé function, we have

$$\gamma_i = \frac{a\alpha_i + b\alpha_i^2}{1 + c\alpha_i} \qquad (15)$$

We selected only 3 "critical" points, which are shown in the Table 2. All of them are from the range, which is not covered by above mentioned properties (9-12).

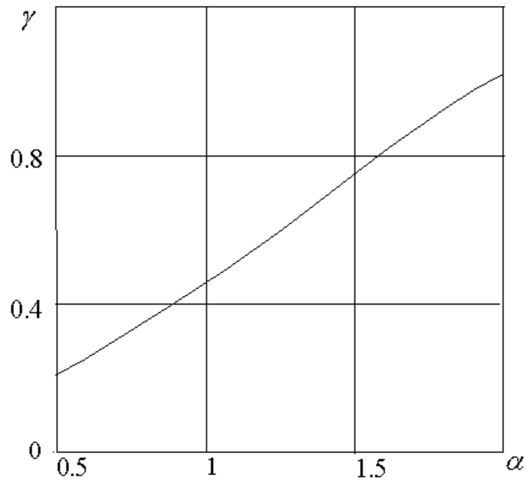
Table 2. Points for approximation of function γ

i	t	q	α	F	F_1	F_0	γ_i
1	5	0,5	0,5	2,450	2,236	3,910	0,1973
2	2	0,5	2	2,862	4	1,883	1,112
3	2	0,5	5	4,318	32	1,703	3,534

The result of solving (15) for these points is the following

$$a = 0{,}3096; b = 0{,}2846; c = 0{,}2909 \qquad (16)$$

The graph of (15) as a function of α is depicted in Figure 3.



Figur 3 - Function γ depending on shape parameter α .

Thus, (8), (9), (10), (13) along with (16) provide the approximate analytical solution for CIF of the g–renewal process. We validated this approximation in interval $0 \le q \le 1{,}1$, $0{,}5 \prec \alpha \le 5$ and $CIF \le 5$ against the Monte Carlo solution with 10^7 trials. The maximum relative error of the approximation is 2,6% in interval $0{,}75 \le \alpha \le 2$. The error increases significantly when $\alpha < 0{,}75$ or $\alpha > 2$, because the assumption that γ in (13) depends only on α is valid only if $0{,}75 \le \alpha \le 2$.

We adjusted (13) to the case $\alpha > 2$ by including dependence γ on all three variables t, α, q. We use polynomial function in the following form:

$$\gamma = G_0 + (\alpha - 2)\left[A_0 + B_0 t + (0.5 - q)(C_0 + D_0 t)\right], \qquad (17)$$

where $G_0 = 1{,}112$ is obtained from the condition that the value of γ obtained by (17) coincides with that obtained by (13) if $\alpha = 2$. To obtain values for coefficients A_0 and B_0, we consider two points for $\alpha = 4$ from Table 2.

Table 2. Points for approximation of function γ if $2 < \alpha \le 5$

i	t	q	α	F	F_1	F_0	γ
1	1,4	0,5	4	1,647	3,842	1,108	2,344
2	2	0,5	4	3,958	16	1,742	2,6855

Using these points, we obtain $A_0 = 0{,}2176,\ B_0 = 0{,}2846$. To identify coefficients C_0 and D_0, we used Monte Carlo results in interval $2 < \alpha \le 5$ and minimized the maximum relative error of approximation (17), which yielded $C_0 = 0{,}09,\ D_0 = -0{,}48$.

We used similar approach in the case when $\alpha < 0.75$ by considering the following polynomial approximation of CIF:

$$\gamma = G_1 + (0{,}75 - \alpha)\left[A_1 + B_1 t + (0{,}5 - q)(C_1 + D_1 t)\right], \qquad (18)$$

where $G_1 = 0{,}3220$ is obtained from the condition that the value of γ obtained by (18) coincides with that obtained by (13) if $\alpha = 0.75$. To obtain values for coefficients A_1 and B_1, we consider 2 points for $\alpha = 0{,}5$ as shown in Table 3.

Table 3. Points for approximation of function γ if $\alpha < 0{,}75$

i	t	q	α	F	F_1	F_0	γ
1	2	0,5	0,5	1,5068	1,4142	2,0491	0,2274
2	15	0,5	0,5	4,4196	3,8729	9,4142	0,1498

Using these points, we obtain $A_1 = -0{,}3302, B_1 = -0{,}02391$. To identify coefficients C_1 and D_1, we used Monte Carlo results in interval $0{,}5 \le \alpha < 0{,}75$ and minimized the maximum relative error of approximation (18), which yielded C_1=0,49, D_1= –0,009. It is important to note that, ultimately, we joined the above three approximation (15, 17, 18) in interval $0{,}5 \le \alpha \le 5$ and obtained a continuous function as a result.

To calculate the maximum relative error of the obtained final approximation (15-18) with respect to the MC solution, we selected 9 points from interval $0{,}5 \le \alpha \le 5$, 12 points from interval $0 < q \le 1{,}1$ and 20 points for the time variable under condition that $CIF \le 5$. In total, 2160 points were used. The results are summarized in Table 4. As it follows from the table, the maximum error is 2,6% in the most practically important intervals $0{,}5 \le \alpha \le 3$, $0 \le q \le 1.1$ and $CIF \le 5$. The upper limit for α– interval can be extended to 5 with the maximum error of 4,3%.

Table 4. Maximum relative error of the g–renewal function with respect to the MC solution for various values of Weibull shape parameter

α	5	4	3	2	0,8	0,75	0,7	0,6	0,5
Max error, %	4,3	3,7	2,5	1,3	1,8	2,6	2,0	1,8	1,9

GRP ESTIMATION USING THE APPROXIMATE CIF SOLUTION

4.1. Estimation procedure

The obtained approximations allow to efficiently solve the "reverse" problem: the estimation of g–renewal process parameters α, λ, q . Kaminskiy & Krivtsov [10] used the MC-based nonlinear least squares estimation and Yañez, et. al [20] followed by Mettas & Zhao [16] used the maximum likelihood estimation of the *g*–renewal process. Our approach is based on the obtained approximations of CIF and an improved strategy of finding the minimum of the sum of residual squares.

At first, we set *q*=1 (under which, CIF is numerically equal to the cumulative hazard function of the underlying Weibull distribution – see [12], and find initial estimates of the shape and the scale parameters using the hazard paper approach [17]. Then, we find the minimum of the sum of residual squares with respect to the shape parameter followed by the scale parameter. The process is repeated iteratively for each selected value of *q* in interval [0,1] until the desired accuracy for all parameters is reached. When q is changed in the loop, we use the estimates of the shape and the scale parameters obtained at the preceding step as the initial values.

4.2 Simulated Data Example

To test the accuracy of the described estimation procedure, we simulated the CIF of the g–renewal process with the following parameters $q = 0,5$, $\alpha = 2,0$ and $1/\lambda = 10$ with 10^7 trials – see Table 5.

The fragment of calculation of residual sum of squares (SS) is represented in Table 6. There is a clear minimum of the residual sum of squares (Set 3), which is reached when the g–renewal process parameters are close to exact values $q = 0,5$, $\alpha = 1,9999$ and $1/\lambda = 9,9707$.

Table 5. Input data generated by Monte Carlo solution

Time	1	2	4	6	8	10	12	14
CIF	0,009973	0,03968	0,1559	0,3405	0,5831	0,8732	1,2032	1,5686
MC SE	5,5E-5	1,1E-4	2,3E-4	3,6E-4	5,0E-4	6,8E-4	8,9E-4	1,1E-3

Table 6. Estimates of g–renewal process parameters based on data in Table5

	Set 1	Set 2	**Set 3**	Set 4	Set 5
q	0,46	0,48	**0,50**	0,52	0,54
$1/\lambda$	9,873	9,921	**9,971**	10,02	10,06
α	1,9999	1,9999	**1,9999**	1,987	1,973
SS	1,26E-04	3,55E-05	**4,69E-06**	9,08E-06	1,78E-05

We would like to emphasize that the running time is extremely short. It took only 0,8 sec to calculate the result with good accuracy on the computer with 2,51 GHz processor and 6,00 GB of RAM. For calculation of Gamma faction in the above formula we used Lanczos approximation [15], whose relative error is $2 \cdot 10^{-10}$.

4.3 Real Data Example

This example shows the practical application of the proposed g–renewal process estimation. We use the Empirical Cumulative Intensity Function (ECIF) estimated from the automotive warranty data [11] – see Table 7.

Table 7. CIF of the g–renewal process based on real data

Time	3	6	9	12	15	18	21	24	27
ECIF	0,03	0,09	0,14	0,24	0,38	0,54	0,70	0,90	1,17

In contrast to the previous example, we do not observe a clear minimum of the residual sum of squares – see Table 8. It fluctuates around $2 \cdot 10^{-3}$ in interval $0,8 < q < 1$. The residual SS shown in the last row of Table 8 illustrates that identified combinations (sets) of GRP parameters fit the data almost equally well. This might be explained by three factors: a) a relatively high random variability in the data causing the departure of the ECIF from the ideal CIF of the g–renewal process, b) the relatively low value of the ECIF (of 1.17), implying a relatively low number of recurrent events, and, as a result, the low accuracy in the estimation of the restoration parameter *q*, and c) a potential inaccuracy of the proposed approximate solution used as the basis of the estimation procedure.

Table 8. Estimates of g–renewal process parameters based on data in Table 7

	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
q	1,0	0,96	0,92	0,88	0,84	0,80
$1/\lambda$	25,07	24,95	24,82	24,68	24,53	24,39
α	1,907	1,917	1,927	1,939	1,95	1,963
SS	1,79E-03	1,84E-03	1,88E-03	1,94E-03	2,00E-03	2,06E-03

To verify the influence of the third factor above we validated results using Monte Carlo method. We calculated CIF for two most "diverse" sets of g–renewal process parameters from Table 8: Set 1 and Set 6. As it follows from Table 9, there is a small deference between the CIF values obtained by Monte Carlo method and both of them are good approximations of ECIF.

Table 9. Comparison with Monte Carlo method (10^7 trials)

Time	3	6	9	12	15	18	21	24	27
ECIF	0,03	0,09	0,14	0,24	0,38	0,54	0,70	0,90	1,17
CIF (Set1)	0,0174	0,0653	0,142	0,245	0,375	0,532	0,713	0,920	1,15
SE (Set 1)	7,6E-5	1,5E-4	2,3E-4	3,2E-4	4,2E-4	5,4E-4	6,7E-4	8,3E-4	1,0E-3
CIF (Set 6)	0,0163	0,0633	0,140	0,245	0,375	0,534	0,716	0,922	1,15
SE (Set 6)	7,3E-5	1,5E-4	2,3E-4	3,1E-4	4,2E-4	5,3E-4	6,6E-4	8,1E-4	9,9E-4

For this real data set, we also tested the accuracy and the computational efficiency of the estimation procedure based on the proposed approximate solution of g–renewal equation versus that based on the MC solution. The MC method results are shown in Table 10. They are very close to those represented in the Table 8. However, the running time was significantly higher: 25 minutes vs. 0.8 sec. If the number of MC trials is decreased to 10^5, the process of finding the minimum of the residual sum of squares diverges because of the lack of accuracy of the MC method. The estimation procedure based on the proposed approximate solution is free from this drawback.

Table 10. Estimates of g–renewal process parameters based on data in Table 7 using MC method

	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
q	1,0	0,96	0,92	0,88	0,84	0,80
$1/\lambda$	25,06	24,96	24,84	24,67	24,57	24,41
α	1,908	1,913	1,921	1,936	1,946	1,960
SS	1,77E-03	1,82E-03	1,87E-03	1,92E-03	1,99E-03	2,07E-03

CONCLUSIONS. In this paper, we proposed an approximate solution to the g–renewal equation for the case of the underlying Weibull distribution. The proposed solution is simple to implement, and it has a practically comparable accuracy as the respective Monte Carlo solution.

We also used the proposed approximate solution as a basis for the non-linear least squares estimation of the GRP parameters. Compared to the MC-based estimation, the proposed estimation is much more computationally efficient and is almost similarly accurate.

Finally, we observed that for real-life data sets (which are subject to statistical noise) one can find *multiple* sets of estimates of the underlying Weibull parameters and the restoration factor that yield practically equal fits to the ECIF. This is to say that in the absence of prior information, it would be difficult to explain whether the difference between two ECIF's is caused by the difference on the underlying distribution parameters or in the restoration factors.

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Аннотация

ПРИБЛИЗИТЕЛЬНОЕ РЕШЕНИЕ УРАВНЕНИЯ G-ВОССТАНОВЛЕНИЯ С БАЗОВЫМ РАСПРЕДЕЛЕНИЕМ ВЕЙБУЛЛА

Кривцов В. В., Евкин А. Ю.

Важной характеристикой g-восстановительного процесса, также представляющей существенный практический интерес, является уравнение g-восстановления, которое характеризует матожидание суммарного числа повторных событий в функции времени. Проблема состоит в том, что уравнение g-восстановления не имеет решения в замкнутом виде, кроме случая экспоненциального базового распределения. Решение Монте-Карло [10] хотя и исчерпывающее, но в вычислительном плане достаточно емкое. В данной статье рассматривается сравнительно простое в реализации (в Excel) приблизительное решение, для случая Вейбулловского базового распределения. Точность предложенного решения не отличается от соответствующего решения Монте-Карло более чем на 2%. Основываясь на предложенном решении, мы также рассмотрели процедуру оценки параметров уравнения g-восстановления.

Анотація

ПРИБЛИЗНЕ РОЗВ'ЯЗАННЯ РІВНЯННЯ G-ВІДНОВЛЕННЯ З БАЗОВИМ РОЗПОДІЛОМ ВЕЙБУЛА

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Важливою характеристикою g-відновного процесу, що представляє істотний практичний інтерес є рівняння g-відновлення, яке характеризує маточікування сумарного числа подій, що повторюються у функції часу. Проблема полягає в тому, що у рівняння g-відновлення немає рішення у замкненому виді, за винятком експоненціального базового розподілу. Рішення Монте-Карло [10] хоча і є вичерпним, але в обчислювальному плані досить ємне. В статті розглядається порівняно просте в реалізації (в Excel) приблизне рішення, для випадку базового розподілення Вейбула. Точність запропонованого рішення не відрізняється від відповідного рішення Монте-Карло більш ніж на 2%. Ґрунтуючись на запропонованому рішенні, ми також розглянули процедуру оцінки параметрів рівняння g-відновлення.