MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

STATE BIOTECHNOLOGICAL UNIVERSITY

Faculty of Mechatronics and Engineering

Department of Physics and Mathematics

ELEMENTS OF ANALYTICAL GEOMETRY. A LINE ON THE PLANE

Guidelines for Students Studying a Course of Mathematics in English

for the first (bachelor) level of Higher Education
Specialty 281 "Public Management and Administration"

Kharkiv
2024

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ELEMENTS OF ANALYTICAL GEOMETRY. A LINE ON THE PLANE: Guidelines for Students Studying a Course of Higher Mathematics in English/ State Biotechnology University; authors: N.V. Smetankina, A.O. Pak, T.O. Sychova. – Kharkiv, 2024. – 21 p.

Basic definitions of Analytical Geometry (a straight line on a plane, second order curves) are given. The connection between different forms of line equations is shown. It is recommended for students studying mathematics in English.

Responsible for the issue: N.V. Smetankina, DSc., Prof.

1 A LINE ON THE PLANE

1.1 Basic Definitions

Let on a plane in the rectangular Cartesian coordinate system xOy some line L is given (Fig. 1.1). Point M belongs to the line L $M(x;y) \in L$.

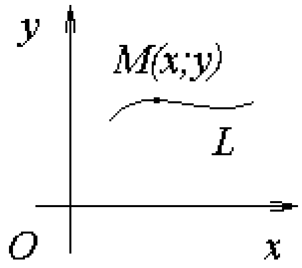

Figure 1.1

Therefore coordinates of the point M are not arbitrary, and related by a certain equation. The equation $L: F(x, y) = 0$ or $y = f(x)$ is called the ***general equation of a line on a plane***.

The elementary line on a plane is a straight line.

1.2 A Straight Line on a Plane

Consider different equations of a straight line.

(1) Equation of a straight line through the point with the normal vector. Let vector $\overline{N} = (A, B)$ is perpendicular to straight line l (Fig. 1.2). This vector is called the ***normal vector*** of a straight line. Formulate an equation of a straight line through the point $M_0(x_0, y_0) \in l$ with the normal vector $\overline{N} = (A, B)$.

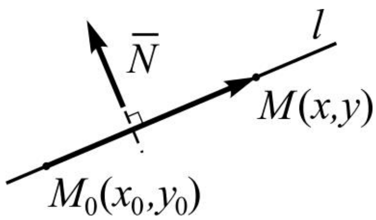

Figure 1.2

We take on a straight line an arbitrary point $M(x, y)$ which is distinct from $M_0(x_0, y_0)$, and write the vector $\overline{M_0M} = (x - x_0, y - y_0)$.

From the condition of perpendicularity of vectors $\overline{N}$ and $\overline{M_0M}$ their scalar product is equal to zero

$$\overline{N} \cdot \overline{M_0M} = 0.$$

From here we obtain ***the equation of a straight line through the point with the normal vector***

$$A(x - x_0) + B(y - y_0) = 0. \qquad (1.1)$$

Remove the parentheses and obtain ***the general equation of a straight line***

$$Ax + By + C = 0, \qquad (1.2)$$

where $C = -Ax_0 - By_0$ is an equation constant term (some number), $A^2 + B^2 \neq 0$.

Now we analyze the general equation (1.2)

1) $C = 0$, $Ax + By = 0$, $y = -\dfrac{B}{A}x$. A straight line passes ***through the origin of coordinates***.

2) $A = 0$, $By + C = 0$, $y = -\dfrac{C}{D} = a \Rightarrow l \parallel Ox$. A straight line is ***parallel to*** X ***-axis.***

3) $B = 0$, $Ax + C = 0$, $x = -\dfrac{C}{A} = b \Rightarrow l \parallel Oy$. A straight ***line is parallel to*** Y***-axis***.

4) $A = C = 0, B \neq 0 \Rightarrow By = 0 \Rightarrow y = 0$ is ***the equation of*** X ***-axis*** .

5) $B = C = 0, A \neq 0 \Rightarrow Ax = 0 \Rightarrow x = 0$ is ***the equation of axis*** Y ***-axis***.

(2) Slope equation of a straight line. Express y from the general equation $Ax + By + C = 0$. Let obtain ***the slope equation of a straight line***

$$y = kx + b, \tag{1.3}$$

where $k = -\frac{A}{B}, \quad b = -\frac{C}{B}$, k is the slope of a straight line, b is the *y*-intercept of a line.

Consider geometrical meaning of k. Let there is a straight line l and a point on straight line $M(x;y) \in l$ (Fig. 1.3).

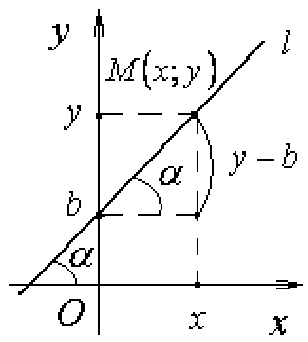

Figure 1.3

From Figure 1.3 we get

$$\tan\alpha = \frac{y-b}{x} = k, \ y - b = kx, \ y = kx + b.$$

Thus, k is ***tangent of the angle between straight line*** l ***and a positive direction of*** X ***-axis***, $k = \tan\alpha$.

Analyze the equation $y = kx + b$ (1.3)

a) Let $k = 0, \quad y = b, \quad \tan\alpha = 0 \Rightarrow \alpha = 0 \Rightarrow l \parallel Ox$.

b) Let $\tan\alpha$ does not exist $\Rightarrow \alpha = \frac{\pi}{2} \Rightarrow \ l \parallel Oy$.

c) Let $b = 0, \quad y = kx$. A straight line passes through an origin of coordinates.

(3) The point-slope equation of a straight line. Let straight line l passes through the point $M_0(x_0; y_0) \in l$. An angle between a straight line and X -axis is $\alpha \neq \frac{\pi}{2}$ and $\tan\alpha = k$.

The slope equation is

$$y = kx + b. \tag{1.4}$$

As point M_0 belongs to a straight line, its coordinates satisfy to the equation (1.4)

$$y_0 = kx_0 + b. \tag{1.5}$$

Subtract from the equation (1.4) the equation (1.5)

$$y - y_0 = k(x - x_0). \tag{1.6}$$

This is the ***point-slope equation.***

As k is an arbitrary number, the equation (1.6) describes a few lines which pass through point M_0. Therefore, the equation (1.6) is called also ***the equation of a pencil of lines.***

(4) Intercept form of the equation of a straight line. Transform a general equation (1.2) (Fig. 1.4)

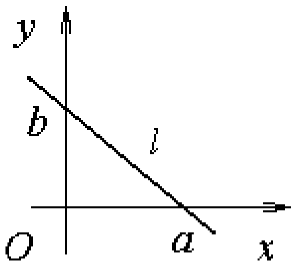

Figure 1.4

$$Ax+By+C=0 \Rightarrow Ax+By=-C \Rightarrow \frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1 \Rightarrow \quad \frac{x}{a}+\frac{y}{b}=1, \qquad (1.7)$$

where $a=-\frac{C}{A}, b=-\frac{C}{B}$ are intercepts by the straight line on the coordinate axes.

The equation (1.7) is ***the intercept form of the equation of a straight line.***

(5) Canonical and parametric equations of a straight line. Let a straight line l passes through point $M_0(x_0; y_0) \in l$ and l is parallel to the vector $\bar{s}=(m,n)$, $l \,||\, \bar{s}$. Vector $\bar{s}$ is called ***directing vector of a straight line***.

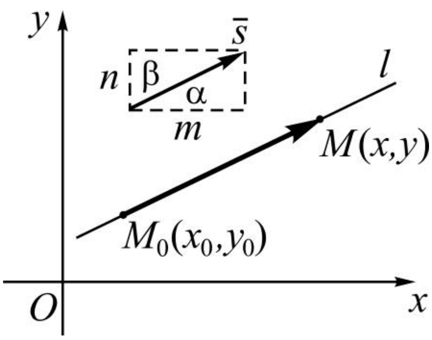

Figure 1.5

Choose on straight line l an arbitrary point $M(x,y)$ and consider a vector $\overline{M_0M}=(x-x_0, y-y_0)$ (Fig. 1.5).

Vectors $\overline{M_0M}$ and $\bar{s}$ are collinear $\overline{M_0M} \,||\, \bar{s}$, therefore their coordinates are proportional

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}. \tag{1.8}$$

We obtained ***the canonical equation of a straight line.***

If in the equation (1.8) we suppose the constant of proportionality equal to some parameter t, then we obtain ***the system of the parametric equations of a straight line***

$$\left(\frac{x-x_0}{m}=t,\ \frac{y-y_0}{n}=t\right) \Rightarrow \begin{cases} x=x_0+mt \\ y=y_0+nt, \quad \forall t \in \mathbf{R}. \end{cases} \tag{1.9}$$

(6) A two-point equation. Formulate the equation of a straight line which passes through points $M_1(x_1,y_1)$ and $M_2(x_2,y_2)$. For this purpose we choose on straight line l point $M(x,y)$ which is distinct from points M_1 and M_2, and form vectors $\overline{M_1M}$ and $\overline{M_1M_2}$ (Fig. 1.6)

$$\overline{M_1M}=(x-x_1, y-y_1), \quad \overline{M_1M_2}=(x_2-x_1, y_2-y_1).$$

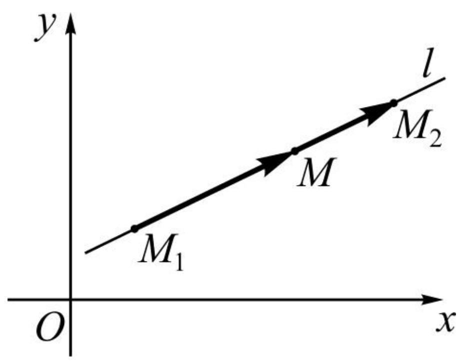

Figure 1.6

These vectors are collinear

$$\overline{M_1M} \,||\, \overline{M_1M_2}\,.$$

Therefore, the required equation of a straight line has the form

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \qquad (1.10)$$

1.3 Primal Problems on a Straight Line

1.1.1 Intersection of Two Straight Lines

Let two straight lines l_1 and l_2 set by general equations (Fig. 1.7)

$$\begin{aligned} &A_1x+B_1y+C_1=0 \ (l_1), \\ &A_2x+B_2y+C_2=0 \ (l_2). \end{aligned} \qquad (1.11)$$

If *straight lines are intersected,* they have one generic point. It means, that system (1.11) has a unique solution.

By the Cramer's Rule we have a ***condition of intersection of two straight lines***

$$\Delta=\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}=A_1B_2-B_1A_2\neq 0 \Rightarrow \frac{A_1}{A_2}\neq\frac{B_1}{B_2}\,. \qquad (1.12)$$

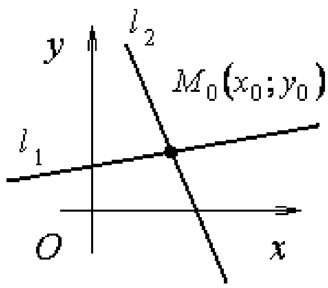

Figure 1.7

Solution of the system of equations (1.11) is $x_0 = \frac{\Delta_1}{\Delta}, \quad y_0 = \frac{\Delta_2}{\Delta}$ (see Fig. 1.7).

Example. Consider ***a problem about a supply and demand***. Find a commodity market equilibrium point. Data are presented in the Table 1.1

Table 1.1

Price, monetary units	2.0	2.5	3.0	3.5	4.0
Demand volume, kg	250	230	210	190	170
Supply volume, kg	200	210	220	230	240

The problem solution is reduced to search of across point of two straight lines. Write equations of supply and demand as two-point equations of straight lines (see (1.10)).

The equation of a straight line of demand has the form

$$\frac{p-2}{4-2} = \frac{q-250}{170-250} \Rightarrow q = -40p + 330.$$

The equation of a straight line of supply

$$\frac{p-2}{4-2} = \frac{s-200}{240-200} \Rightarrow s = 20p + 160.$$

We solve the system of two equations under the condition $q = s$ (1.8)

$$\begin{cases} q = -40p + 330 \\ s = 20p + 160 \end{cases} \Rightarrow \begin{cases} p^* = 2.8; \\ q^* = 183.3. \end{cases}$$

So, we have obtained equilibrium price $p^* = 2.8$ and equilibrium demand volume $q^* = 183.3$.

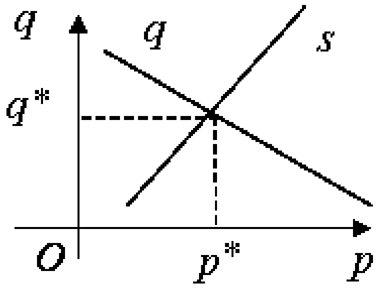

Figure 1.8

At $p = 2.5$ demand for the goods is $q = -40 \cdot 2.5 + 330 = 230$. It exceeds supply $s = 20 \cdot 2.5 + 160 = 210$. So, in the market we have a commodity deficiency.

1.1.2 Angle between Two Straight Lines

Let define an angle between straight lines which are given by the slope equations (Fig. 1.9)

$$y = k_1 x + b_1 (l_1)\ ,\ k_1 = \tan \varphi_1,$$

$$y = k_2 x + b_2 (l_2)\ ,\ k_2 = \tan \varphi_2 .$$

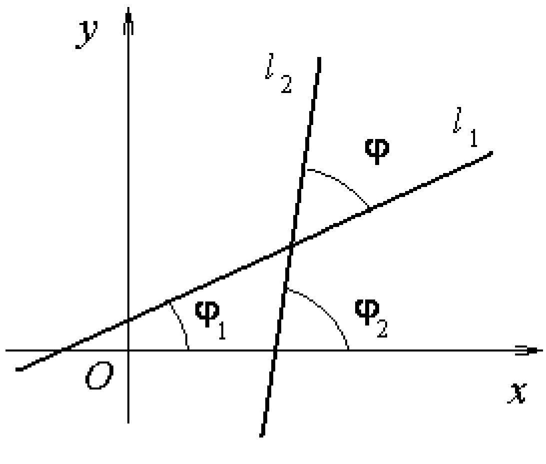

Figure 1.9

Then the ***angle between straight lines*** can be calculated under the formula

$$\tan\varphi = \tan(\varphi_2 - \varphi_1) \Rightarrow \frac{\tan\varphi_2 - \tan\varphi_1}{1 + \tan\varphi_2 \cdot \tan\varphi_1},\ \tan\varphi = \frac{k_2 - k_1}{1 + k_2 \cdot k_1},$$

$$\varphi = \operatorname{arctan}\varphi = \frac{k_2 - k_1}{1 + k_2 \cdot k_1}. \qquad (1.13)$$

From here we have a ***condition of parallelism of straight lines***. If $l_1 \,||\, l_2$, then $\varphi = 0 \Rightarrow \tan\varphi = 0 \Rightarrow k_2 - k_1 = 0 \Rightarrow$

$$k_1 = k_2.$$

Let

$$l_1 \perp l_2 \Leftrightarrow \varphi = \pi/2 \Leftrightarrow \cot\varphi = 0 \Leftrightarrow \cot\varphi = \frac{1}{\tan\varphi} = \frac{1 + k_2 \cdot k_1}{k_2 - k_1} = 0 \Rightarrow$$

$$\Rightarrow 1 + k_2 \cdot k_1 = 0 \Rightarrow k_2 = -\frac{1}{k_1}.$$

It follows a ***condition of perpendicularity of straight lines***

$$k_2 = -\frac{1}{k_1}.$$

1.1.3 Distance from a Point to a Straight Line

Let the straight line $Ax + By + C = 0$ (l) and point $M_0(x_0, y_0) \notin l$ are given (see Fig. 1.10). Distance d from point $M_0(x_0, y_0)$ to a straight line l is the length of perpendicular $d = |M_0N|$ which is calculated under the formula

$$d = \left| \frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}} \right|. \qquad (1.14)$$

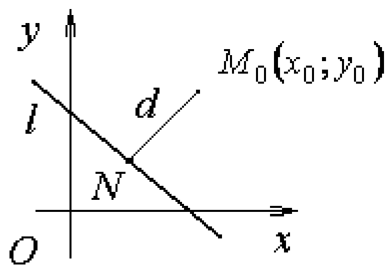

Figure 1.10

Example. Find a distance from a point $M_0(2,7)$ to a straight line $3x+7y-9=0$.

Solution.

$$d=\left|\frac{3\cdot 2+7\cdot 7-9}{\sqrt{3^2+7^2}}\right|=\frac{46}{\sqrt{58}}\approx 6.0 \text{ (linear units).}$$

2 SECOND ORDER CURVES

2.1 Circle

Curve of the second order on a plane is called a set of points which coordinates are of the same system as the Cartesian coordinates satisfy the following equation

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0, \qquad (2.1)$$

where A, B, C, D, E and F are real numbers; $A^2+B^2+C^2\neq 0$.

Equation (2.1) is called the ***general equation of the second order curves***.

If $A=C=1$, and $B=0$, we have the ***general equation of circle***

$$x^2+y^2+Dx+Ey+F=0. \qquad (2.2)$$

Definition. ***Circle*** is a geometrical place of points equidistant from a certain point called centre.

A distance from any point of a circle to the centre is called radius. If we simplify the equation (2.2), we obtain ***the canonical equation of a circle***

$$(x-x_0)^2+(y-y_0)^2=R^2, \qquad (2.3)$$

where the point $C(x_0,y_0)$ is center of a circle. (Fig. 2.1)

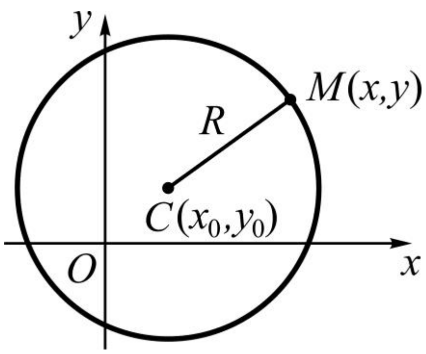

Figure 2.1

If the centre lays in an origin of coordinates, the canonical equation has the form

$$x^2+y^2=R^2. \qquad (2.4)$$

2.2 Ellipse

Definition. An ***ellipse*** is the set of points in a plane whose distances from two fixed points in the plane have a constant sum (Fig. 2.2).

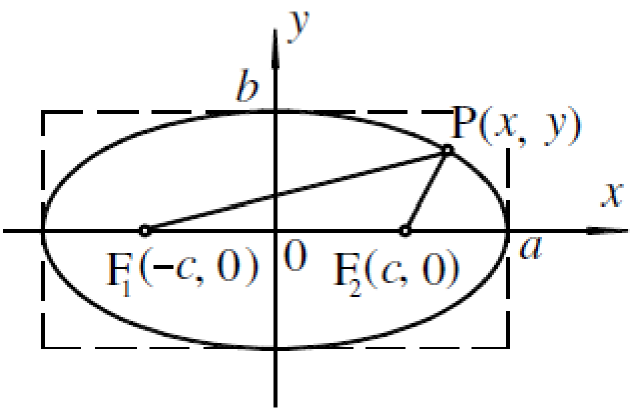

Figure 2.2

The two fixed points are the ***foci*** of the ellipse. The line through the foci of an ellipse is the ***ellipse's focal axis***. The point of the axis halfway between the foci is the ellipse's ***center***. The points where the focal axis crosses the ellipse are the ellipse's ***vertices***.

If the foci are $F_1(-c,0)$ and $F_2(c,0)$, and the sum of the distances $PF_1 + PF_2$ is denoted by $2a$, then the coordinates of a point P on the ellipse satisfy the equation

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical and square again, and we obtain ***the canonical equation of an ellipse***

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \Rightarrow$$

$$\Rightarrow 4a\sqrt{(x+c)^2 + y^2} = 4a^2 + 4cx \Rightarrow$$

$$\Rightarrow a^2\left(x^2 + 2cx + c^2 + y^2\right) = a^4 + 2a^2cx + c^2x^2 \Rightarrow$$

$$\Rightarrow a^2x^2 + a^2c^2 + a^2y^2 = a^4 + c^2x^2 \Rightarrow$$

$$\Rightarrow \left(a^2 - c^2\right)x^2 + a^2y^2 = a^2\left(a^2 - c^2\right) \Rightarrow \left|a^2 - c^2 = b^2\right| \Rightarrow$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (2.5)$$

The ***major axis*** of the ellipse described by equation (2.5) is the line segment of length $2a$ joining the points $(a,0)$ and $(-a,0)$. The ***minor axis*** of the ellipse (2.5) is the line segment of length $2b$ joining the points $(0,b)$ and $(0,-b)$. The number a itself is called the ***semimajor axis*** and the number b the ***semiminor axis***. The number c, which can be found as $c = \sqrt{a^2 - b^2}$ is the ***center-to-focus distance*** of the ellipse.

If we keep a fixed and vary c over the interval $0 \le c \le a$ the resulting ellipses will vary in shape. They are circles if $c = 0$ (so that $a = b$) and flatten as c increases.

In the extreme case $c = a$, the foci and vertices overlap and the ellipse degenerates into a line segment. We use the ratio $\frac{c}{a}$ to describe the various shapes the ellipse can take. We call this ratio the ellipse’s eccentricity.

Definition. The ***eccentricity*** of the ellipse is the number

$$\varepsilon = \frac{c}{a}, \quad 0 < \varepsilon < 1, \qquad (2.6)$$

or

$$\varepsilon = \frac{c}{a} \Rightarrow \left| c = \sqrt{a^2 - b^2} \right| \Rightarrow \left(\varepsilon = \frac{1}{a}\sqrt{a^2 - b^2},\ \varepsilon^2 = 1 - \left(\frac{b}{a}\right)^2 \right).$$

2.3 Hyperbolas

Definition. A ***hyperbola*** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference.

The two fixed points $F_1(-c, 0)$ and $F_2(c, 0)$ are the ***foci*** of the hyperbola (Fig. 2.3).

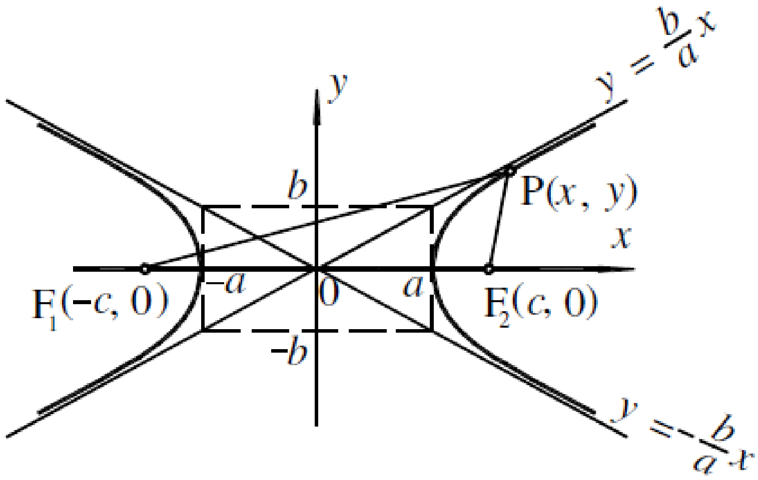

Figure 2.3

Hyperbolas have two branches. For points on the right-hand branch of the hyperbola shown here $PF_1 - PF_2 = 2a$; for points on the left-hand branch, $PF_2 - PF_1 = 2a$. If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ and the constant difference is $2a$, then a point $P(x, y)$ lies on the hyperbola if and only if

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a,$$

or

$$\sqrt{(x+c)^2+y^2}=\sqrt{(x-c)^2+y^2}\pm 2a.$$

We simplify this equation and get ***the canonical equation of a hyperbola***

$$(x+c)^2+y^2=(x-c)^2+y^2\pm 4a\sqrt{(x-c)^2+y^2}+4a^2 \Rightarrow$$

$$\Rightarrow 4cx-4a^2=\pm 4a\sqrt{(x-c)^2+y^2} \Rightarrow$$

$$\Rightarrow c^2x^2-2a^2cx+a^4=a^2\left(x^2-2cx+c^2+y^2\right)\Rightarrow$$

$$\Rightarrow \left(c^2-a^2\right)x^2-a^2y^2=a^2\left(c^2-a^2\right)\Rightarrow \left| c^2-a^2=b^2 \right| \Rightarrow$$

$$\Rightarrow b^2x^2-a^2y^2=a^2b^2 \Rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \quad (2.7)$$

2.4 Parabola

Definition. A ***parabola*** is the set of points in a plane that are equidistant from a given fixed point and fixed line in this plane.

The fixed point is the parabola's ***focus*** $F(0,p)$. The fixed line is the parabola's ***directrix*** $y=-p$ (Fig. 2.4).

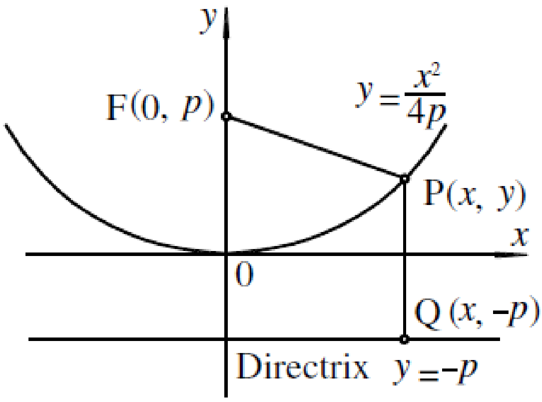

Figure 2.4

A point $P(x,y)$ lies on the parabola if and only if $PF=PQ$. From the distance formula,

$$PF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}\ ,$$

$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{(y+p)^2}\ .$$

When we equate these expressions, square and simplify, we get the ***canonical equation of a parabola***

$$x^2 = 4py\ . \tag{2.8}$$

This equation reveals the parabola's symmetry about the *y*-axis (2.8). We call the *y*-axis the ***axis of the parabola*** (short for "axis of symmetry").

The point where a parabola crosses its axis, midway between the focus and directrix, is called the ***vertex of a parabola***. The vertex of the parabola $y^2 = 4px$ lies at the origin. The number p is the focal length of the parabola, and $4p$ is the ***width of the parabola at the focus***.

The chief application of parabolas involves their use as reflectors of light and radio waves.

Educational Edition

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