MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

STATE BIOTECHNOLOGICAL UNIVERSITY

Faculty of Mechatronics and Engineering

Department of Physics and Mathematics

ELEMENTS OF VECTOR ALGEBRA

Guidelines for Students Studying a Course of Mathematics in English

for the first (bachelor) level of Higher Education
Specialty 281 "Public Management and Administration"

Kharkiv
2023

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ELEMENTS OF VECTOR ALGEBRA: Guidelines for Students Studying a Course of Higher Mathematics in English/ State Biotechnology University; authors: N.V. Smetankina, A.O. Pak, T.O. Sychova. – Kharkiv, 2023. – 25 p.

Basic definitions of Vector Algebra, linear operations on vectors, concepts of scalar, vector and mixed product of vectors are given. Guidelines contain tasks for independent work. It is recommended for students studying mathematics in English.

Responsible for the issue: N.V. Smetankina, DSc., Prof.

1 VECTORS

1.1 Basic Definitions

There are two aspects of quantities: ***scalar*** and ***vector*** ones. Quantity which is defined by a numerical value is called as a ***scalar***. For example, a mass, volume, temperature, goods cost. Quantity which is characterized by numerical value and a direction is called as ***vector*** quantity. For example, velocity, acceleration, force.

Definition. ***Vector*** is a directed line segment (Figure 1).

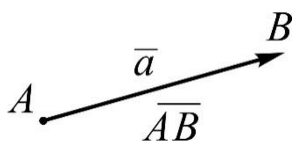


Figure 1

Point A is the initial point; point B is the terminal (end) point.

$$\overline{AB} = \overline{a}.$$

If the vector has no application point, it is ***a free vector***. We shall consider free vectors.

Definition. **The length** or **modulus** of a vector $\overline{AB}$ is called the length of the segment and is denoted by $|\overline{a}|$, $|\overline{AB}|$.

A vector with zero length is called a ***zero vector*** $\overline{O}$,

$$|\overline{O}| = 0.$$

The vector with unit length is called a ***unit vector*** $\bar{e}$,

$$|\bar{e}| = 1.$$

A unit vector with direction vector $\bar{a}$ is called an ***ort*** of vector $\bar{a}$. Denote $\bar{a}^0$.

Definition. Vectors lying on the same straight line or on the parallel straight lines are called ***collinear*.**

Vectors $\bar{a}, \bar{b}, \bar{c}$ are collinear ones, $\bar{a} \parallel \bar{b} \parallel \bar{c}$ (Figure 2).

Figure 2

Definition. Vectors lying on the same plane or on parallel planes are called ***coplanar*.**

Vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are the coplanar vectors (Figure 3).

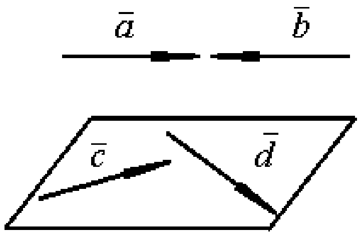
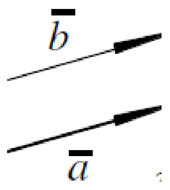


Figure 3

Definition. The vectors $\bar{a}$ and $\bar{b}$ are called ***equal,*** if they are *collinear* and have the *same length* and *direction*.

The vectors $\bar{a}$ and $\bar{b}$ on Figure 4 are equal (Figure 4).



$$\begin{cases} |a| = |b|, \\ \bar{a} \,|\,|\, \bar{b}, \\ \bar{a} \uparrow\uparrow \bar{b} \end{cases}$$

Figure 4

1.2 Linear Operations

(A) Sum of vectors

Triangle rule. Let $\bar{a}$ and $\bar{b}$ be the given free vectors. Draw the vector $\bar{b}$ from the terminal point of the vector $\bar{a}$. ***The sum of vectors*** $\bar{a}$ and $\bar{b}$ is the vector $\bar{c} = \bar{a} + \bar{b}$ connecting the initial point of the vector $\bar{a}$ with the terminal point of the vector $\bar{b}$ provided that the initial point of the vector $\bar{b}$ coincides with the terminal point of the vector $\bar{a}$. (Figure 5)

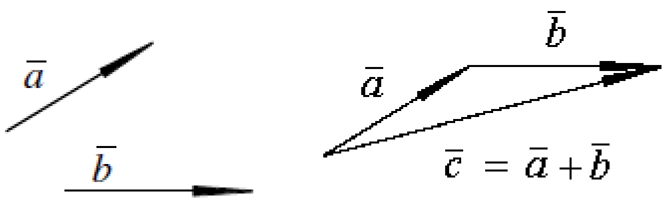


Figure 5

Parallelogram Rule. ***The sum of vectors*** $\bar{a}$ and $\bar{b}$ is the vector $\bar{c}$. It is a diagonal of the parallelogram constructed on vectors $\bar{a}$ and $\bar{b}$ (Figure 6).

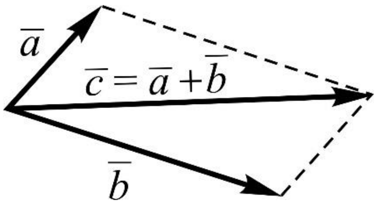


Figure 6

A sum of any number of vectors can be calculated according to the polygon rule.

Polygon Rule. The ***sum of several vectors*** is a vector which connecting the initial point of the first vector with the terminal point of last vector.

For example, the sum of vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ is the vector (Figure 7)

$$\vec{S} = \vec{a} + \vec{b} + \vec{c} + \vec{d} .$$

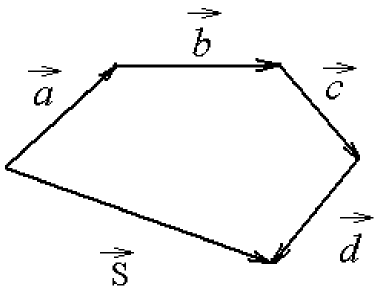


Figure 7

The Properties of the Sum of Vectors

1) $\bar{a}+\bar{b}=\bar{b}+\bar{a}$,

2) $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}=\vec{a}+\vec{b}+\vec{c}$;

3) $\vec{a}+\left(-\vec{a}\right)=\vec{0}$,

4) $\vec{a}+\vec{0}=\vec{a}$.

(B) Difference of Vectors

The difference of vectors $\bar{a}$ and $\bar{b}$ is the sum of vector $\bar{a}$ with a vector which is opposite to vector $\bar{b}$ (Figure 8)

$$\bar{d}=\bar{a}-\bar{b},\ \bar{d}=\bar{a}+\left(-\bar{b}\right).$$

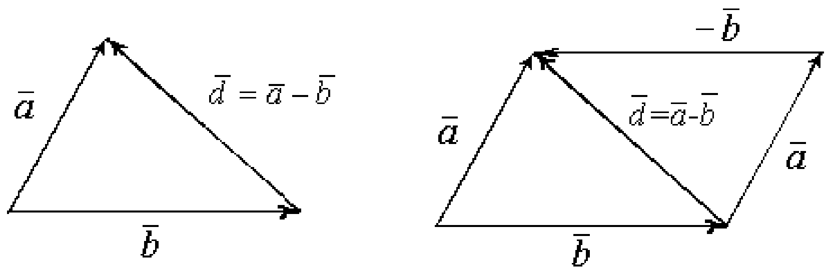


Figure 8

(C) Product of a Vector by a Scalar

Let the vector $\bar{a}$ and some number λ are given, $\lambda \neq 0$. ***The product of a vector*** $\bar{a}$ ***by a scalar (number)*** λ is the vector $\bar{b}$ with modulus $|\lambda|\cdot|\bar{a}|$. If $\lambda > 0$, its direction coincides with direction $\bar{a}$; if $\lambda < 0$, it is opposite to direction $\bar{a}$ (Figure 9),

$$\bar{b}=\lambda\bar{a},\ |\bar{b}|=|\lambda|\cdot|\bar{a}|.$$

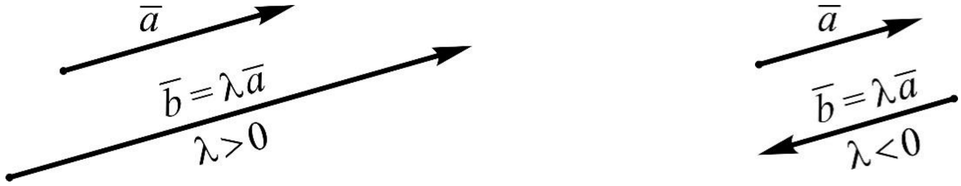


Figure 9

Such equalities are true

$$1\cdot\vec{a}=\vec{a};\ (-1)\vec{a}=-\vec{a};\ 0\cdot\overline{a}=\overline{a}.$$

Any vector can be presented as

$$\overline{a}=|\overline{a}|\cdot\overline{a}^{0},$$

where $\overline{a}^{0}$ is an unit vector (ort).

It follows ***condition of collinearity*** of vectors $\overline{a}$ and $\overline{b}$

$$\overline{b}=\lambda\cdot\overline{a}.$$

Collinear vectors are proportional.

Example. Vectors $\overline{a}=(1,3,-2),\overline{b}=(1,2,0),\overline{c}=(2,3,6)$ are given. Find their linear combination $\overline{d}=2\overline{a}+3\overline{b}-\overline{c}$.

Solution.

$$\overline{d}=2(1,3,-2)+3(1,2,0)-(2,3,6)=(3,9,-10).$$

1.3 Projection of the vector onto an axis

Let vectors $\overline{a}$ and $\overline{b}$ are given (Figure 10). The angle between vectors $\overline{a}$ and $\overline{b}$ is the least angle φ on which we turn one of vectors before its coincidence to another $0\le\varphi<\pi$.

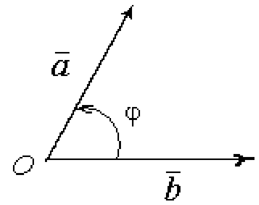


Figure 10

Let a vector $\overline{AB}$ and an axis $\bar{l}$ are given (Figure 11). On axis l the point A has coordinate x_1, and the point B has coordinate x_2.

Difference $x_2 - x_1$ is called the ***projection of vector*** $\overline{AB}$ onto the axis l. Denote $\text{Pr}_l\,\overline{AB} = x_2 - x_1$.

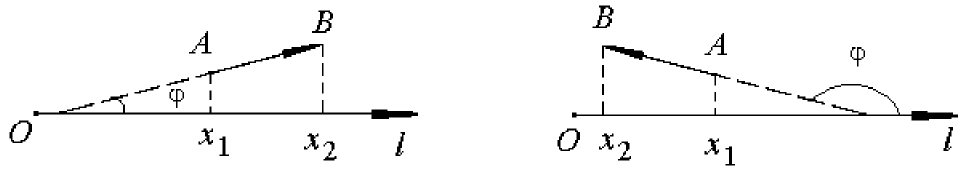


Figure 11

If $x_2 > x_1$, the angle φ between a vector and an axis is *acute*, if $x_2 < x_1$ an angle is *obtuse*.

Let consider the basic theorems about projections.

Theorem. The projection of vector $\overline{AB}$ onto the axis l is equal to product of vector length on cosine of angle φ between the vector and the axis

$$\text{Pr}_l\,\overline{AB} = \left|\overline{AB}\right| \cdot \cos\varphi. \qquad (1)$$

Theorem. The projection of the sum of two vectors onto an axis is equal to the sum of their projections onto this axis

$$\Pr_l \overline{AC} = \Pr_l \overline{AB} + \Pr_l \overline{BC}.$$

Theorem. If the vector multiplies by a number, its projection multiplies by this number too

$$\Pr_l\left(\lambda \cdot \overline{AB}\right) = \lambda \cdot \Pr_l\left(\overline{AB}\right).$$

1.4 Coordinates of a Vector

Consider the rectangular coordinate system (Figure 12). The coordinate axes: Ox is the *X*-axis, Oy is the *Y*-axis, Oz is the *Z*-axis. Let $\bar{i}, \bar{j}, \bar{k}$ be the unit and orthogonal vectors (orts) giving the direction of the coordinate axes Ox, Oy, Oz:

$$|\bar{i}| = |\bar{j}| = |\bar{k}| = 1, \ \bar{i} \perp \bar{j} \perp \bar{k}.$$

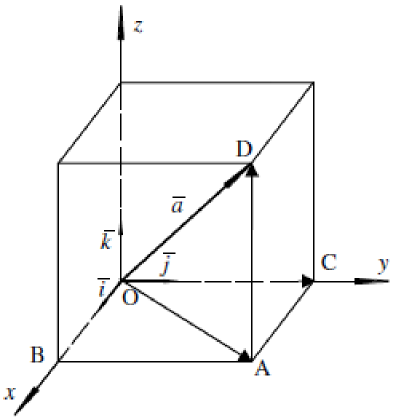


Figure 12

Construct the vector $\bar{a}=\overline{OD}$ and find its projection onto the coordinate axes

$$\bar{a}=\overline{OD}=\overline{OA}+\overline{AD}=\overline{OB}+\overline{OC}+\overline{AD}=x\bar{i}+y\bar{j}+z\bar{k},$$

where

$$x=\mathrm{Pr}_{Ox}\,\bar{a}=a_x,\ y=\mathrm{Pr}_{Oy}\,\bar{a}=a_y,\ z=\mathrm{Pr}_{Oz}\,\bar{a}=a_z.$$

Numbers a_x, a_y, a_z are called the ***coordinates*** of the vector $\bar{a}$. There is the *coordinate form* of the vector $\bar{a}$

$$\bar{a}=\left(a_x, a_y, a_z\right). \tag{2}$$

The *vector form* of the vector $\bar{a}$ or the ***expansion of the vector*** $\bar{a}$ through the *base* $\bar{i}, \bar{j}, \bar{k}$ is

$$\bar{a}=a_x\bar{i}+a_y\bar{j}+a_z\bar{k}. \tag{3}$$

Let the vector is given by the initial point $A(x_1, y_1, z_1)$ and end point $B(x_2, y_2, z_2)$. Then

$$a_x=x_2-x_1,\ a_y=y_2-y_1,\ a_z=z_2-z_1.$$

If vectors $\bar{a}=\left(a_x, a_y, a_z\right)$, $\bar{b}=\left(b_x, b_y, b_z\right)$ are given, we can write through coordinates:

(a) length of the vector $\bar{a}$

$$|\bar{a}|=\sqrt{a_x^2+a_y^2+a_z^2},$$

$$|\bar{a}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}; \tag{4}$$

(b) sum (difference) of vectors

$$\bar{a}\pm\bar{b}=\left(a_x\pm b_x, a_y\pm b_y, a_z\pm b_z\right);$$

(c) product of a vector by a scalar

$$\lambda\bar{a} = (\lambda a_x, \lambda a_y, \lambda a_z).$$

(d) condition of collinearity of vectors $\bar{a}$ and $\bar{b}$

$$\bar{a} \| \bar{b} \Leftrightarrow \frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}. \qquad (5)$$

Consider a vector $\bar{a}$ (Figure 13).

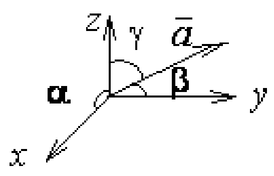


Figure 13

Angles α, β, γ are called the ***direction angles*** of vector $\bar{a}$. Let find projections of the vector onto coordinate axes

$$a_x = |\bar{a}|\cos\alpha, a_y = |\bar{a}|\cos\beta, a_z = |\bar{a}|\cos\gamma.$$

From here

$$\cos\alpha = \frac{a_x}{|\bar{a}|}, \cos\beta = \frac{a_y}{|\bar{a}|}, \cos\gamma = \frac{a_z}{|\bar{a}|}. \qquad (6)$$

are ***direction cosines*** of a vector. Thus the identity is true

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \qquad (7)$$

A unit vector of a vector $\bar{a}$ is defined as

$$\bar{a}^0 = (\cos\alpha, \cos\beta, \cos\gamma).$$

Example. The points $A(1;2;3)$ and $B(3;-4;6)$ are given.

Find a) vector $\overline{AB}$ and its length;

b) direction cosines;

c) a unit vector (ort) of $\overline{AB}$.

Solution.

a) Denote $\overline{AB} = \overline{a}$. Calculate coordinates of the vector:

$$a_x = x_2 - x_1;\ a_y = y_2 - y_1;\ a_z = z_2 - z_1,$$

$$a_x = 3 - 1 = 2;\ a_y = (-4) - 2 = -6;\ a_z = 6 - 3 = 3,$$

$$\overline{a} = 2\overline{i} - 6\overline{j} + 3\overline{k}.$$

The modulus of vector is calculated by the formula

$$|\overline{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}, \quad |\overline{AB}| = \sqrt{2^2 + (-6)^2 + 3^2} = \sqrt{49} = 7.$$

b) Calculate direction cosines of the vector

$$\cos\alpha = \frac{a_x}{|\overline{a}|} = \frac{2}{7}, \cos\beta = \frac{a_y}{|\overline{a}|} = -\frac{6}{7}, \cos\gamma = \frac{a_z}{|\overline{a}|} = \frac{3}{7}.$$

Check up this result

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \quad \left(\frac{2}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 + \left(\frac{3}{7}\right)^2 = 1, \quad 1 = 1.$$

c) Ort of the vector is

$$\overline{a}^0 = \frac{\overline{a}}{|\overline{a}|} = \frac{2\overline{i} - 6\overline{j} + 3\overline{k}}{7} = \frac{2}{7}\overline{i} - \frac{6}{7}\overline{j} + \frac{3}{7}\overline{k}.$$

2 SCALAR, VECTOR AND MIXED PRODUCTS OF VECTORS

2.1. The Scalar Product

Definition. The ***scalar product*** (dot product) $\overline{a}\cdot\overline{b}$ of vectors $\overline{a}$ and $\overline{b}$ is the number (scalar) equal to the product of their moduli on cosine of the angle between them $\varphi = \overline{a}\,\hat{}\,\overline{b}$ (Figure 14)

$$\overline{a}\cdot\overline{b} = |\overline{a}|\cdot|\overline{b}|\cdot\cos\varphi. \tag{8}$$

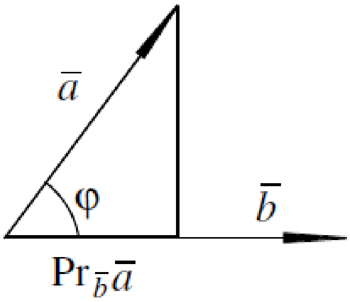


Figure 14

Let write scalar product in another way

$$\overline{a}\cdot\overline{b} = |\overline{a}|\cdot \mathrm{Pr}_{\overline{a}}\,\overline{b} = |\overline{b}|\cdot \mathrm{Pr}_{\overline{b}}\,\overline{a}.$$

Consider properties of the scalar product.

Properties of the Scalar Product

1) $\overline{a}\cdot\overline{b} = \overline{b}\cdot\overline{a}$,

2) $\overline{a}\cdot(\overline{b}+\overline{c}) = \overline{a}\cdot\overline{b}+\overline{a}\cdot\overline{c}$,

3) $(\lambda\bar{a})\cdot\bar{b}=\lambda(\bar{a}\cdot\bar{b})=\bar{a}\cdot(\lambda\bar{b})$,

4) $\bar{a}\cdot\bar{a}=|\bar{a}|\cdot|\bar{a}|\cdot\cos(\bar{a},\bar{a})=|\bar{a}|^2$ (this is scalar square),

$\bar{i}\cdot\bar{i}=1,\ \bar{j}\cdot\bar{j}=1,\ \bar{k}\cdot\bar{k}=1$,

5) if $\bar{a}\cdot\bar{b}=0$, then $\bar{a}\perp\bar{b}$,

$\bar{i}\cdot\bar{j}=\bar{j}\cdot\bar{k}=\bar{k}\cdot\bar{i}=0$.

If the scalar product of two non-zero vectors equals zero, then they are ***perpendicular (orthogonal)***.

If vectors are given by coordinates

$$\bar{a}=a_x\bar{i}+a_y\bar{j}+a_z\bar{k},\ \bar{b}=b_x\bar{i}+b_y\bar{j}+b_z\bar{k},$$

then scalar product is equal to the sum of products of coordinates

$$\bar{a}\cdot\bar{b}=a_xb_x+a_yb_y+a_zb_z. \qquad (9)$$

Obtain **the *angle between vectors*** $\bar{a}$ and $\bar{b}$ from definition

$$\bar{a}\cdot\bar{b}=|\bar{a}|\cdot|\bar{b}|\cdot\cos\varphi \Rightarrow \cos\varphi=\frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\cdot|\bar{b}|} \Rightarrow$$

$$\Rightarrow \cos\varphi=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}\cdot\sqrt{b_x^2+b_y^2+b_z^2}} \Rightarrow \varphi=\arccos(\cos\varphi). \qquad (10)$$

Condition of perpendicularity (orthogonality) of two vectors given in coordinate form is

$$\bar{a}\perp\bar{b} \Leftrightarrow \varphi=\pi/2 \Leftrightarrow \cos\varphi=0 \Leftrightarrow a_xb_x+a_yb_y+a_zb_z=0. \qquad (11)$$

Mechanical meaning of scalar product: scalar product is the work A. Work is equal to product of magnitude of force $\bar{F}$ on magnitude of translation $\bar{S}$ and on the angle cosine between their directions $A = \bar{F} \cdot \bar{S} = |\bar{F}| \cdot |\bar{S}| \cdot \cos(\bar{F}, \bar{S})$.

Example. Vectors $\bar{a} = 3\bar{i} + 4\bar{j} + 7\bar{k}$ and $\bar{b} = 6\bar{i} - 3\bar{j} + 2\bar{k}$ are given.

Find $(\bar{a} + \bar{b})^2$.

Solution. $(\bar{a} + \bar{b})^2 = \bar{a}^2 + 2\bar{a}\bar{b} + \bar{b}^2$; $\bar{a}^2 = |\bar{a}|^2 = 3^2 + 4^2 + 7^2 = 74$;

$\bar{b}^2 = |\bar{b}|^2 = 6^2 + (-3)^2 + 2^2 = 49$; $\bar{a} \cdot \bar{b} = 3 \cdot 6 + 4 \cdot (-3) + 7 \cdot 2 = 20$;

$(\bar{a} + \bar{b})^2 = 74 + 2 \cdot 20 + 49 = 163.$

2.2 The Vector Product

Definition. The ***vector product*** of the vectors $\bar{a} = (a_x, a_y, a_z)$ and $\bar{b} = (b_x, b_y, b_z)$ is the vector $\bar{c} = \bar{a} \times \bar{b}$ defined as follows :

(a) the modulus of the vector is equal to the area of parallelogram constructed on the vectors $\bar{a}$ and $\bar{b}$,

$$|\bar{c}| = S = |\bar{a}| \cdot |\bar{b}| \cdot \sin\varphi, \quad \varphi = \left(\bar{a}, \hat{\ } \bar{b}\right); \qquad (12)$$

b) vector $\bar{c}$ is perpendicular $\bar{a}$ and $\bar{b}$, $\bar{c} \perp \bar{a}$, $\bar{c} \perp \bar{b}$;

c) vector $\bar{c}$ forms with the ordered pair vectors $\bar{a}$ and $\bar{b}$ the right triple of vectors.

We have a right triple of vectors if the shortest rotation of the first vector to the second one is observed from the end point of the third vector in the counterclockwise (Figure 15).

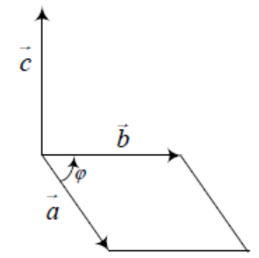


Figure 15

Properties of the Vector Product

1) $\bar{a}\times\bar{b}=-\left(\bar{b}\times\bar{a}\right)$ (the vector product is anticommutative);

2) $\lambda\left(\bar{a}\times\bar{b}\right)=\left(\lambda\bar{a}\right)\times\bar{b}=\bar{a}\times\left(\lambda\bar{b}\right)$;

3) $\bar{a}\times\left(\bar{b}+\bar{c}\right)=\bar{a}\times\bar{b}+\bar{a}\times\bar{c}$;

4) $\bar{a}\times\bar{b}=\bar{0}\Leftrightarrow\bar{a}\,||\,\bar{b}$, $\bar{a}\neq 0$, $\bar{b}\neq 0$ (***condition of collinearity of two vectors***).

The vector product of unit vectors (orts) are

$$\bar{i}\times\bar{i}=\bar{j}\times\bar{j}=\bar{k}\times\bar{k}=0;$$

$$\bar{i}\times\bar{j}=\bar{k},\quad \bar{j}\times\bar{k}=\bar{i},\quad \bar{k}\times\bar{i}=\bar{j};$$

$$\bar{j}\times\bar{i}=-\bar{k},\quad \bar{k}\times\bar{j}=-\bar{i},\quad \bar{i}\times\bar{k}=-\bar{j}.$$

Write the coordinate form of the vector product for vectors $\bar{a}=\left(a_x,a_y,a_z\right)$ and $\bar{b}=\left(b_x,b_y,b_z\right)$.

$$\bar{a}\times\bar{b}=\left(a_x\bar{i}+a_y\bar{j}+a_z\bar{k}\right)\times\left(b_x\bar{i}+b_y\bar{j}+b_z\bar{k}\right)=a_xb_y\left(\bar{i}\times\bar{j}\right)+$$

$$+a_xb_z\left(\bar{i}\times\bar{k}\right)+a_yb_x\left(\bar{j}\times\bar{i}\right)+a_yb_z\left(\bar{j}\times\bar{k}\right)+a_zb_x\left(\bar{k}\times\bar{i}\right)+a_zb_y\left(\bar{k}\times\bar{j}\right)=$$

$$=a_xb_y\bar{k}-a_xb_z\bar{j}-a_yb_x\bar{k}+a_yb_z\bar{i}+a_zb_x\bar{j}-a_zb_y\bar{i}=$$

$$=\left(a_yb_z-a_zb_y\right)\bar{i}-\left(a_xb_z-a_zb_x\right)\bar{j}+\left(a_xb_y-a_yb_x\right)\bar{k};$$

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\cdot\bar{i}-\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}\cdot\bar{j}+\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\cdot\bar{k}. \qquad (13)$$

Mechanical meaning of vector product: vector product is a torque of force concerning a point.

2.3 The Mixed Product

Definition. ***The mixed product*** (***triple scalar product***) of three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is called the vector product of two from them multiplied as scalar on the third vector

$$\left[\bar{a}\times\bar{b}\right]\cdot\bar{c} \text{ or } \bar{a}\cdot\bar{b}\cdot\bar{c}.$$

The mixed product has certain geometrical meaning. The modulus of the mixed product is equal to the volume of the parallelepiped constructed on the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$

$$V=\left|\left[\bar{a}\times\bar{b}\right]\cdot\bar{c}\right|. \qquad (14)$$

Write the *coordinate form* of the mixed product for vectors $\bar{a}=\left(a_x,a_y,a_z\right)$, $\bar{b}=\left(b_x,b_y,b_z\right)$ and $\bar{c}=\left(c_x,c_y,c_z\right)$

$$\left[\overline{a}\times\overline{b}\right]\cdot\overline{c}=\begin{vmatrix} a_x & a_y & a_z \\ a_x & a_y & a_z \\ a_x & a_y & a_z \end{vmatrix}. \quad (15)$$

Let remind that vectors lying on the same plane or on parallel planes are called coplanar. Now formulate ***condition of coplanarity of three vectors***. Nonzero vectors are coplanar, if their mixed product is equal to zero:

$$\overline{a}\neq 0,\ \overline{b}\neq 0,\ \overline{c}\neq 0,\ \left[\overline{a}\times\overline{b}\right]\cdot\overline{c}=0 \Leftrightarrow \overline{a},\ \overline{b},\ \overline{c}\ \text{are coplanar;}$$

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=0. \quad (16)$$

3 TASK "ELEMENTS OF VECTOR ALGEBRA"

Let there be given coordinates of vertices of a pyramid *ABCD*.

<u>Find:</u>

a) Vectors $\overline{AB}$, $\overline{AC}$, $\overline{AD}$ and their modules,

b) The angle between vectors $\overline{AB}$ and $\overline{AC}$;

c) Projection of vector $\overline{AD}$ onto vector $\overline{AB}$;

d) Area of side *ABC*;

e) Volume of a pyramid *ABCD*.

Data for completing an individual task

1. *A*(2; -3; 1), *B*(6; 1; -1), *C*(4; 8;-9), *D*(2; -1; 2).
2. *A*(5; -1; -4), *B*(9; 3; -6), *C*(7; 10;-14), *D*(5; 1; -3).
3. *A*(1;-4 ; 0), *B*(5; 0; -2), *C*(3; 7;-10), *D*(1; -2; 1).
4. *A*(-3; -6; 2), *B*(1; -2; 0), *C*(-1; 5; -8), *D*(-3; -4; 3).
5. *A*(-1; 1; -5), *B*(3; 5; -7), *C*(1; 12; -15), *D*(-1; 3; -4).
6. *A*(-4; 2; -1), *B*(0; 6; -3), *C*(-2; 13; -11), *D*(-4; 4; 0).
7. *A*(0; 4; 3), *B*(4; 8; 1), *C*(2; 15; -7), *D*(0; 6; 4).
8. *A*(-2; 0; -2), *B*(2; 4; -4), *C*(0; 11; -12), *D*(-2; 2; -1).
9. *A*(3; 3; -3), *B*(7; 7; -5), *C*(5; 14; -13), *D*(3; 5; -2).
10. *A*(4; -2; 5), *B*(8; 2; 3), *C*(6; 9; -5), *D*(4; 0; 6).

Sample Task

Let there be given coordinates of vertices of a pyramid *ABCD*:

A(2; 1; 0), *B*(3; -1; 2), *C*(13; 3; 10), *D*(0; 1; 4).

<u>Find:</u>

a) Vectors $\overline{AB}$, $\overline{AC}$, $\overline{AD}$ and their modules,

b) The angle between vectors $\overline{AB}$ and $\overline{AC}$;

c) Projection of vector $\overline{AD}$ onto vector $\overline{AB}$;

d) Area of side *ABC*;

e) Volume of a pyramid *ABCD*.

Solution

1. A free vector $\bar{a}$ can be presented by the expansion corresponding to the coordinate axes or to the orts

$$\bar{a} = a_x\bar{i} + a_y\bar{j} + a_z\bar{k}\ , \tag{T.1}$$

where a_x, a_y, a_z are vector projections on the axes $0x$, $0y$ i $0z$.

If points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ are given, then projections of vector $\bar{a} = \overline{M_1M_2}$ onto coordinate axes can be found by formulas

$$a_x = x_2 - x_1;\ a_y = y_2 - y_1;\ a_z = z_2 - z_1. \tag{T.2}$$

Let's substitute coordinates of points *A* and *B* to (T.2) and (T.1) and we obtain vector $\overline{AB}$

$$\overline{AB} = \bar{i} - 2\bar{j} + 2\bar{k} .$$

Similarly, we find

$$\overline{AC} = 11\bar{i} + 2\bar{j} + 10\bar{k}; \quad \overline{AD} = -2\bar{i} + 4\bar{k} .$$

The modulus of vector $\bar{a}$ given by the formula (T.1) is calculated by the formula

$$|\bar{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} . \tag{T.3}$$

Applying (T.3), we obtain modules of the found vectors

$$\left|\overline{AB}\right| = 3, \quad \left|\overline{AC}\right| = 15, \quad \left|\overline{AD}\right| = 2\sqrt{5} .$$

2. Cosine of the angle between the vectors is the scalar product of these vectors divided by the product of their modules. Find the scalar product of vectors $\overline{AB}$ and $\overline{AC}$

$$\overline{AB} \cdot \overline{AC} = 1 \cdot 11 + (-2) \cdot 2 + 2 \cdot 10 = 27 .$$

Hence,

$$\cos(\overline{AB}\,\widehat{,}\,\overline{AC}) = \overline{AB} \cdot \overline{AC} \Big/ \left(\left|\overline{AB}\right| \cdot \left|\overline{AC}\right|\right) = 27/(3 \cdot 15) = 3/5 = 0{,}6; \quad (\overline{AB}\,\widehat{,}\,\overline{AC}) = 53^0 .$$

3. The projection of the vector $\overline{AD}$ onto the vector $\overline{AB}$ is equal to the scalar product of these vectors divided by the modulus of the vector $\overline{AB}$:

$$\Pr_{\overline{AB}} \overline{AD} = \overline{AB} \cdot \overline{AD} \Big/ \left|\overline{AB}\right| = (1 \cdot (-2) + (-2) \cdot 0 + 2 \cdot 4)/3 = 2 .$$

4. The area of side ABC is equal to half of area of the parallelogram constructed on vectors $\overline{AB}$ and $\overline{AC}$. The area of the parallelogram constructed on vectors $\overline{AB}$ and $\overline{AC}$ is equaled to the modulus of the vector product of these vectors:

$$\overline{AB}\times\overline{AC}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -2 & 2\\ 11 & 2 & 10\end{vmatrix}=-24\bar{i}+12\bar{j}+24\bar{k};\quad S=\left|\overline{AB}\times\overline{AC}\right|=36 \text{ (square units).}$$

Then the area of side ABC is equal to

$$S_{ABC}=\frac{1}{2}S=18 \text{ (square units).}$$

5. The volume of the parallelepiped constructed on three not non-coplanar vectors, is equaled to the modulus of their mixed product. Calculate the mixed product $\left(\overline{AB}\times\overline{AC}\right)\cdot\overline{AD}$

$$V_{par}=\left|\left(\overline{AB}\times\overline{AC}\right)\cdot\overline{AD}\right|=\begin{vmatrix}1 & -2 & 2\\ 11 & 2 & 10\\ -2 & 0 & 4\end{vmatrix}=144 \text{ (cubic units).}$$

Thus, the parallelepiped volume is equal to 144 cubic units, and the volume of the given pyramid $ABCD$ is equal to one sixth of the volume of parallelepiped:

$$V_{pyr}=\frac{1}{6}V_{par}=\frac{1}{6}\cdot 144=24 \text{ (cubic units).}$$

Educational Edition

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