UDC 519.95

ESTIMATION OF G–RENEWAL PROCESS PARAMETERS AS AN ILL–POSED INVERSE PROBLEM

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Statistical estimation of g–renewal process parameters have been considered by many authors. We show that this inverse problem is mathematically incorrect (the solution is not unique and/or is sensitive to statistical error) and requires Tikhonov's regularization. Regardless of the estimation method, the respective objective function usually involves parameters of the underlying life–time distribution and simultaneously the restoration parameter. In this paper, we propose to regularize this inverse problem by separating the estimation of the aforementioned parameters. Using a simulation study, we show that the resulting extrapolation accuracy of the proposed method is considerably higher than that of the existing methods.

Acronyms

CDF – cumulative distribution function
CIF – cumulative intensity function
ECIF – empirical cumulative intensity function
GRP – generalized renewal (or g–renewal) process
HPP – homogeneous Poison process
IFR – increasing failure rate
MC – Monte Carlo
MTTF – mean time to failure
NHPP – non–homogeneous Poison process
ORP – ordinary renewal process
PDF – probability density function
RSS – residual sum of squares
SE – standard error
TTF – time to the first failure

Notation

q – restoration (or repair effectiveness) factor
$\vec{\alpha}$ – vector of parameters of the underlying life–time distribution
t – time
$W(t)$– g–renewal function denoting the expected cumulative number of events (failures)
$f(t)$ – probability density function
$F(t)$ – cumulative distribution function
λ, α – respectively, the scale and the shape parameters of Weibull distribution

1. Introduction

We consider a practically important problem of the estimation of the g–renewal process parameters and the prediction of g–renewal function, based on restricted (in time) empirical data. This problem often arises, for example, in maintenance optimization (defining maintenance/ replacement time of a system) [12,13] and in forecasting of warranty repairs/costs [14,15].

From the standpoint of mathematics, statistical estimation, i.e., evaluating model's parameters based on the data can be viewed as an *inverse* problem. This is in contrast to a *forward* problem, which involves evaluating/predicting data points based on the model parameters. Evans and Stark [5] draw some formal parallels between statistical estimation problems and mathematical inverse problems. For example, they point out that *identifiability* (distinct models yield distinct probability distributions for the observed data) is similar to *uniqueness* (the forward operator maps at most one model into the observed data). Further, *consistency* (model parameters can be estimated with arbitrary accuracy as the number of data points grow) is related to *stability of recovery* (small changes in the data produce small changes in the recovered model).

Most of the inverse problems are considered to be mathematically incorrect and/or ill–posed. A typical remedy in this case is the so–called *regularization*, i.e., the introduction of additional information in order to solve an ill–posed problem or to prevent model overfitting. Bayesian estimation, Ridge regression and Lasso regression are all examples of regularization in statistical science.

In this paper, we show that the problem of estimating parameters of a g–renewal process is mathematically ill–posed or incorrect (ill–conditioned). We propose a regularization approach, which is neither Bayesian, nor Ridge/Lasso regression related. It is based on separating the underlying distribution parameters from the GRP restoration factor in their estimation. Using a simulation study, we show that the resulting extrapolation accuracy of the proposed method is considerably higher than that of the existing methods.

The paper is structured as follows. In Section II, we overview the g–renewal process and represent it as an inverse problem. In Section III, we discuss existing estimation methods of the g–renewal equation parameters and propose a regularization approach along with the respective estimation procedure. We use a simulation study to compare the accuracy of the proposed approach relative to the existing methods. In Section IV, we use a practical case study to show the efficiency of the proposed method.

2. G–renewal estimation as an inverse problem

Originally introduced by Kijima & Summita [6], the generalized renewal (g–renewal) process gained its *practical* popularity only after methods for estimating its parameters had become available. The non–linear least square estimation of the g–renewal process was first offered by Kaminskiy & Krivtsov [1]. The maximum likelihood procedures were subsequently discussed by Yañez, et. al [2] and Mettas & Zhao [3]. The estimation of the g–renewal restoration factor was addressed in detail by Kahle & Love [4].

Mathematically, estimation of the g–renewal process amounts to solving the following g–renewal equation with respect to its parameters:

$$W(t)=\int_0^t\left(g(\tau\,|\,0)+\int_0^\tau w(x)g(\tau-x\,|\,x)dx\right)d\tau, \quad (1)$$

where

$$g(t\,|\,x)=\frac{f(t+qx,\vec{\alpha})}{1-F(qx,\vec{\alpha})},\quad t,x\geq 0;\quad w(x)=\frac{dW(x)}{dx};$$

and $F(t)$ and $f(t)$ are the *cumulative distribution function* (CDF) and *probability density function* (PDF) of the underlying failure time distribution (note that $g(t|0) = f(t)$); q is the restoration factor, and $\vec{\alpha}$ is the vector of parameters of the underlying life–time distribution.

In other words, Equation (1) needs to be solved with respect to q and $\vec{\alpha}$:

$$\{\vec{\alpha},q\}=\arg\left\langle W(t)=\int_0^t\left(g(\tau\,|\,0)+\right.\right.$$
$$\left.\left.+\int_0^\tau w(x)\frac{f(\tau-x+qx,\vec{\alpha})}{1-F(qx,\vec{\alpha})}dx\right)d\tau\right\rangle$$

It can be shown that this inverse problem is ill–posed or incorrect. According to Hadamard [7], a *well–posed problem* is such for which: a) a solution exist, b) the solution is unique, c) the solution depends continuously on the data in some reasonable topology. A *correct problem* has almost the same definition except for c) the solution must be stable (meaning that small statistical errors in the data should not much influence the solution).

Consider a g–renewal process with arbitrary and infinitely increasing cumulative intensity function, *W(t)*, corresponding to some underlying failure–time distribution function, *F(t)*, and the restoration factor $q\neq 1$. In this setting, one can derive another solution corresponding to q=1: $F(t)=1-e^{-W(t)}$, simply because the cumulative intensity function, *W(t)*, of the NHPP (i.e., g–renewal process with q=1) is formally equal to the cumulative hazard, *H(t)*, of the respective underlying failure–time distribution [9]. This is to say that for a g–renewal process with *any* value of $q\neq 1$, one can *always* find a solution in the above form of *F(t)* and q=1. Hence, the solution is not unique in general case, and, therefore, the respective (inverse) problem is ill–posed.

Even the *ordinary* renewal process (q=0) can be shown to have an ill–posed (or ill–conditioned) reverse problem. Recall that the ordinary renewal equation involves a convolution integral (which, of course, is also present in Eq. 1) :

$$W(t)=F(t)+\int_0^t F(t-\tau)dW(\tau) \quad (2)$$

As such, the respective inverse problem can be considered as a *deconvolution* problem, which is typically ill–conditioned. Its solution is not stable with respect to the calculation error and/or empirical noise [10, 11].

The reverse (estimation) problem becomes even more complicated for the g–renewal process, as in addition to the parameters of the underlying life–time distribution, one has to also deal with the restoration parameter. When discussing the estimation of the g–renewal process with the underlying Weibull distribution in [8], we noticed that two "competing" vectors of significantly different g–renewal parameters yielded practically indistinguishable values of the CIF at several time cross–sections.

As an illustration, let us choose the class of Weibull distribution functions, $F(t)=1-\exp(-\lambda t)^{\alpha}$, with scale parameter, λ, and shape parameter, α as the underlying failure–time distribution of a g–renewal process. Now, consider Figure 1, where we show three CIF's simulated (under n=10^7 trials) with three sets of the underlying parameters: Case 1: $\{\lambda_1$=1.0, α_1=2.0, q_1=0$\}$, Case 2: $\{\lambda_2$=0.949, α_2=1.675, q_2=0$\}$, Case 3: $\{\lambda_3$=0.949, α_3=1.675, q_3=0.1$\}$. Cases 2 and 3 can be considered as empirical data fluctuating close to the exact solution (Case 1).

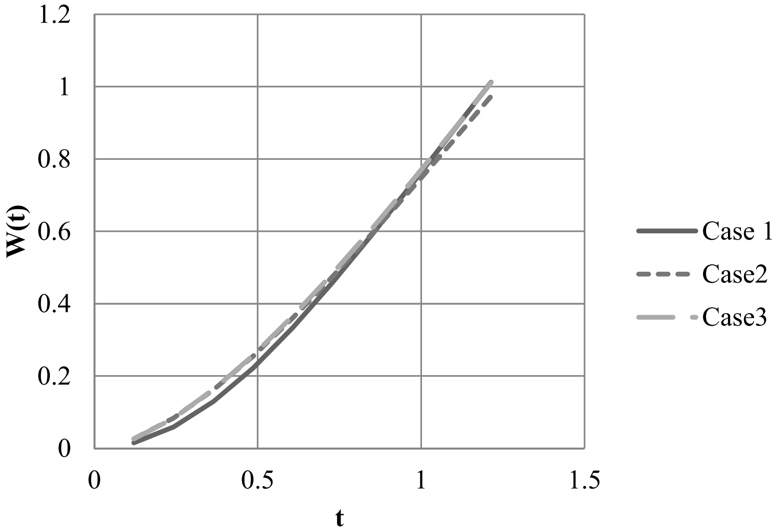

Figure 1 – CIF's simulated (n=10^7) under various sets of Weibull parameters:
Case 1: $\{\lambda_1$=1.0, α_1=2.0$\}$, Case 2: $\{\lambda_2$=0.949, α_2=1.675$\}$, Case 3: $\{\lambda_3$=0.949, α_3=1.675, q=0.1$\}$

Note that with as much as 20% difference in the Weibull shape parameter between Cases 1 and 2, the maximum difference in the respective values of the CIF is around 4% (at t=1.214). Moreover, even though all cases presented in Fig. 1 can be considered as good approximations (interpolations) in the interval $0 \le W(t) \le 1$, they have significantly different g–renewal process parameters and, as a consequence, yield significantly different extrapolations of the g–renewal function, as shown in Fig. 2. This indicates that even in the class of Weibull distribution functions, the inverse problem is ill–conditioned.

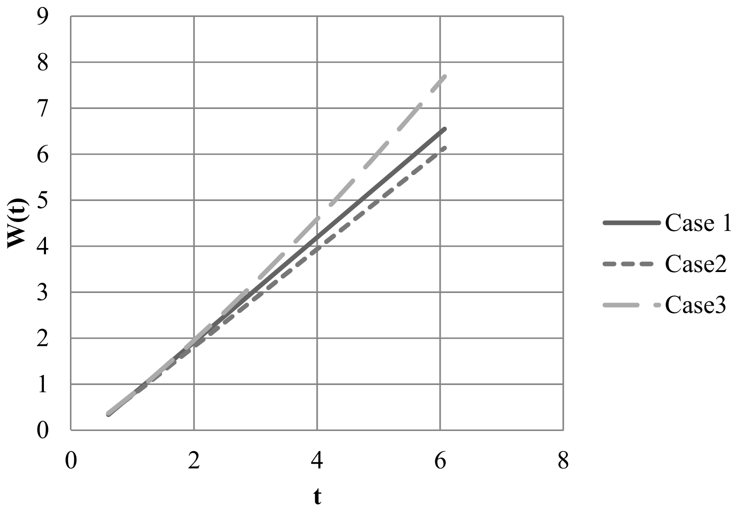

Figure 2 – CIF's from Figure 1 extrapolated to 6 units in time

There are many developed general methods of regularization of inverse problem [10], however all of them require significant amount of additional calculations, because typically another optimization parameter (e.g., a Lagrange multiplier) is introduced in the problem. These methods are efficient in the case when corresponding forward problem is easy to solve. In our case, the forward problem is described by integral equation (1), which does not have a closed form solution. In this paper, we try different methods for solving the problem of extrapolation (prediction) of the g–renewal function and finally select a method, which is relatively stable with respect to a statistical error.

3. Accuracy of g–renewal function extrapolation

Below we consider four different ways of approximating (estimating) the g–renewal function (Eq. 1) and compare them from the standpoint of the estimation and, ultimately, the extrapolation accuracy.

In our numerical experiments, we simulated the g–renewal process to obtain input data for further estimation. First, the "exact" CIF, was obtained via a Monte Carlo simulation (n=10^7) using the underlying Weibull distribution with the shape parameter of α=2, the scale parameter of λ=1 and various values of the restoration factor q = 0, 0.5, 1. The trial data set, representing the *empirical* CIF that simulates "statistical noise", was obtained via a Monte Carlo simulation (n=400) with the same set of parameters as in the exact CIF. The range of simulated $W(t)$ was extended to approximately a unity thus allowing for (some proportion of) recurrent events. Finally, in all four methods, we used the sum of least squares as the objective function for minimization; however, the likelihood function or any other objective function can be used as well.

3.1. Methodology

Method 1: Generalized Pade approach

In this approach, to approximate $W(t)$, we use the generalized Pade functions in the following form:

$$W(t) = \frac{a_0 + a_1 t + a_2 t^2 + ... + a_m t^n}{1 + b_1 t + b_2 t^2 + ... + b_k t^k}, \tag{3}$$

where a_i , b_i – model parameters estimated from the data.

Pade functions have the following advantages: a) they have the same or better convergence compared to the power series, b) they can include functions with singularities, c) estimation of the Pade function coefficients involves solving *only linear* equations. While looking for the best approximation, we try different values of m and k, and various substitutions for variable t, such as t^p, e^{pt}, ($-1 \le p \le 1$), and $log(t)$.

Method 2: G–Renewal MC approach

This method, originally proposed in [1] and further improved in [16], searches for the optimal values of λ, α and q to minimize the objective function (in this case, RSS) by simulating trial values of the g–renewal function and comparing them to the exact or empirical CIF.

$$\{\lambda, \alpha, q\} = \arg\left\langle \min\left(W(t) - W(t \mid \lambda, \alpha, q)\right)^2 \right\rangle \tag{4}$$

It must be noted that relative to Method 1, in Method 2 the inverse problem is better defined, because simulations are performed in the framework of Eq. 1 and under the assumption that the underlying function is Weibull, that is, additional information about the g–renewal process is used.

Method 3: G–Renewal MC approach with consecutive estimation

Existing methods of g–renewal process estimation [2–4], whether LSQ– or MLE–based, in their objective functions, usually involve parameters of the underlying life–time distribution *simultaneously* with the restoration parameter. Method 2 is one of such examples.

In Method 3, we propose to *separate* the estimation of the underlying life–time distribution parameters and the restoration parameter in two *consecutive* steps. The former can be readily estimated from times to the first failure (TTF), for which the respective statistical procedures are well established. The latter can then be estimated from the entire data set involving the first and the repeat failures. In the second step, the objective function optimization is done with respect to just *one* parameter instead of *three* parameters (for the considered case of the Weibull distribution) thus increasing residual degrees of freedom and reducing the complexity of calculations.

From the standpoint of the inverse problem solving, Method 3 is "more regularized" than Method 2, let alone Method 1. This is because it uses information not only about the framework of the g–renewal process, but also about the underlying life–time distribution, which is an explicit term of the g–renewal equation (see Eq. 1).

Method 4: G–Renewal MC approach with consecutive estimation when TTF distribution is not available
In some settings, such a single repairable system with multiple consecutive failures, TTF distribution is not available or cannot be properly estimated due to the lack of statistical data. However, the TTF distribution *can* be estimated from the empirical CIF, when the value of this CIF is truncated in such a way that it corresponds *primarily* to the first failures. In this case, the following approximation can be used for estimating Weibull parameters:

$$(\lambda t)^{\alpha} \approx W(t) \tag{5}$$

The sensitivity of the solution relative to the choice of the truncation point is discussed in Section 3.2.

3.2. Comparative Analysis of Estimation Accuracy

In this section, we compare the estimation accuracy of the four methods for various values of the g–renewal restoration parameter q = 0, 0.5, and 1. The underlying TTF distribution is Weibull with the shape and the scale parameters of 2 and 1, respectively, as discussed above. The estimating is done based on the empirical CIF (n=400).

The results are shown in the Tables 1–3. The tables are structured as follows. The first column is the time interval. The second column is the "exact" CIF obtained by the Monte Carlo method with 10^7 trials. The third column is the "empirical" CIF with 400 trials, which is used for estimating parameters in all 4 methods. The relative error with respect to the exact solution is estimated by the corresponding RSS, which is shown in the last row of the third column. The fourth column is the percent of first failures (PFF) in relative units; note that PFF is used in Method 3 only for estimating the underlying Weibull distribution parameters. The remaining columns represent the CIF evaluated by Monte Carlo simulation after estimating its parameters by the corresponding methods. RSS values calculated for each method with respect to the exact and the empirical data are shown in the last two rows of these columns. We also calculated the overall relative error of the g–renewal parameters estimation as follows:

$$\varepsilon_i = \sqrt{\frac{\left(\alpha-\hat{\alpha}\right)^2+\left(\lambda-\hat{\lambda}\right)^2+\left(q-\hat{q}\right)^2}{\alpha^2+\lambda^2+q^2}} \tag{6}$$

where $\hat{\alpha}$, $\hat{\lambda}$, $\hat{q}$ are the g–renewal process parameters estimated by various methods.

Restoration Parameter q=0 (Ordinary Renewal Process)

Table 1 summarizes the estimated CIF for the four methods in question along with the respective RSS values relative to both exact and empirical CIF's under q=0, which is the case of the ORP.

Method 1 yielded the following Pade approximation: $W(t) = -0.259t^{0.6}+1.023t^{1.2}$. Method 2 resulted in these estimates of the g–renewal parameters: $\hat{q}$ = 0.020, $\hat{\lambda}$=0.949, $\hat{\alpha}$ = 1.675, ε = 0.147. Method 3 resulted in the following estimates: $\hat{q}$ = 0, $\hat{\lambda}$=1.037, $\hat{\alpha}$ = 2.006, ε=0.017. In this method, the Weibull parameters are estimated from the PFF (the 3rd column in Table 1), and then the restoration factor is estimated from the entire data set in the framework of the g–renewal process equations. Note that we obtained a much more accurate estimation of all g–renewal process parameters applying this method.

In Method 4, the empirical data were truncated at $W(t)$=0.5 for estimating Weibull parameters using (5), and then the restoration factor was calculated. Method 4 resulted in the following estimates of the g–renewal parameters: $\hat{q}$=0, $\hat{\lambda}$=1.057, $\hat{\alpha}$=2.056. With the truncation point of $W(t)$=0.6, the estimated g–renewal parameters are more accurate: $\hat{q}$=0, $\hat{\lambda}$=1.009, $\hat{\alpha}$=1.998, ε=0.004. (The respective column for Method 4 in Table 1 corresponds to the truncation point of $W(t)$=0.6). We noticed that the obtained results significantly depend on the truncation point, because formula (5) is limited by a small time interval and its extension may lead to an approximation error, which is greater than statistical one. On the other hand, the statistical error can be reduced by involving more data through the increase of the time interval. Ultimately, we decided to select truncation point that delivers the minimal value to the objective function (RSS).

Method 2 showed the best approximation of the empirical CIF data (RSS=0.002); however, the obtained values of corresponding g–renewal process parameters ($\hat{q}$=0.020, $\hat{\lambda}$=1.054, $\hat{\alpha}$=1.675) are far from exact values. Comparing all values of RSS in the last row of Table 1, one can conclude that all estimated models are in the same range of accuracy as the empirical data. The same conclusion can be derived from the direct comparison of data from the table. Methods 3 and 4 yield good approximation for the g–renewal process parameters.

Table 1 – Estimated CIF along with the respective RSS values for exact and empirical CIF's under q=0

t	$n=10^7$	n=400	PFF	Method 1	Method 2	Method 3	Method 4
0.061	0.004	0.003	0.003	-0.013	0.008	0.004	0.004
0.121	0.015	0.020	0.020	0.008	0.027	0.016	0.015
0.182	0.033	0.035	0.035	0.039	0.052	0.035	0.033
0.243	0.058	0.058	0.058	0.076	0.084	0.062	0.059
0.304	0.089	0.113	0.110	0.118	0.121	0.095	0.091
0.364	0.127	0.178	0.173	0.163	0.162	0.135	0.129
0.425	0.170	0.213	0.203	0.211	0.207	0.181	0.173
0.486	0.218	0.263	0.248	0.262	0.256	0.233	0.222
0.546	0.271	0.318	0.293	0.315	0.307	0.289	0.275
0.607	0.328	0.370	0.335	0.370	0.362	0.349	0.334
0.668	0.388	0.428	0.388	0.427	0.418	0.412	0.394
0.728	0.451	0.473	0.425	0.485	0.476	0.479	0.458
0.789	0.516	0.535	0.468	0.545	0.536	0.547	0.524
0.850	0.583	0.593	0.510	0.607	0.597	0.617	0.591
0.911	0.651	0.658	0.568	0.669	0.660	0.689	0.660
0.971	0.720	0.720	0.615	0.733	0.723	0.761	0.730
1.032	0.790	0.775	0.650	0.798	0.787	0.833	0.800
1.093	0.860	0.843	0.690	0.865	0.851	0.906	0.871
1.153	0.930	0.918	0.728	0.932	0.916	0.979	0.942
1.214	1.000	1.000	0.778	1.000	0.982	1.052	1.013
RSS (relative to n=400)		n/a		0.003	0.002	0.023	0.013
RSS (relative to $n=10^7$)		0.014		0.015	0.011	0.017	0.001

Restoration Parameter q=0.5 (G–Renewal Process)

Table 2 summarizes the estimated CIF for the four methods in question along with the respective RSS values for both the exact and the empirical CIF's under q=0.5.

Method 1 yielded the following Pade approximation: $W(t) = 5.580 - 11.147\, e^{-0.5t} + 5.566\, e^{-t}$.

Table 2 – Estimated CIF along with the respective RSS values for exact and empirical CIF's under q=0.5

t	$n=10^7$	n=400	PFF	Method 1	Method 2	Method 3	Method 4
0.054	0.003	0.003	0.003	0.004	0.004	0.003	0.003
0.108	0.012	0.013	0.013	0.016	0.014	0.011	0.011
0.162	0.026	0.023	0.023	0.035	0.031	0.025	0.025
0.216	0.046	0.038	0.035	0.060	0.054	0.045	0.045
0.270	0.072	0.063	0.060	0.091	0.083	0.070	0.072
0.324	0.103	0.128	0.120	0.127	0.118	0.101	0.104
0.378	0.140	0.163	0.148	0.168	0.157	0.137	0.142
0.432	0.181	0.203	0.180	0.213	0.201	0.178	0.186
0.486	0.227	0.250	0.220	0.262	0.250	0.225	0.234
0.540	0.278	0.315	0.275	0.315	0.303	0.276	0.287
0.594	0.334	0.363	0.308	0.371	0.359	0.331	0.345
0.648	0.393	0.415	0.353	0.430	0.418	0.391	0.406
0.702	0.457	0.483	0.405	0.492	0.481	0.455	0.470
0.756	0.525	0.543	0.443	0.556	0.545	0.522	0.538
0.810	0.596	0.610	0.478	0.622	0.612	0.594	0.607
0.864	0.670	0.668	0.525	0.690	0.680	0.668	0.679
0.918	0.748	0.745	0.570	0.759	0.750	0.745	0.752
0.972	0.829	0.810	0.595	0.830	0.821	0.826	0.827
1.026	0.913	0.883	0.618	0.902	0.893	0.909	0.903
1.080	0.999	0.975	0.668	0.975	0.967	0.994	0.980
RSS (relative to n=400)		n/a		0.004	0.001	0.008	0.004
RSS (relative to $n=10^7$)		0.010		0.012	0.006	0.0002	0.002

Method 2 resulted in these estimates of the g–renewal parameters: $\hat{q}$=0.14, $\hat{\lambda}$=1.054, $\hat{\alpha}$=1.964, ε=0.16. The value of restoration parameter is far from the exact one. Method 3 showed: $\hat{q}$=0.44, $\hat{\lambda}$=1.01, $\hat{\alpha}$=2.037, ε =0.031 – the most accurate estimates in this case. Method 4 yielded the following: $\hat{q}$ = 0.16, $\hat{\lambda}$=1.07, $\hat{\alpha}$ = 2.105, ε =0.158.

All four methods performed reasonably well. They are in the same range of accuracy as empirical data compared to exact one. Clear winners are Methods 3 and 4. Method 3 is the absolute winner (RSS=0.00015). Most importantly, it yields the best approximation for the g–renewal parameters.

Restoration Parameter q=1 (Non–Homogeneous Poisson Process)

Table 3 summarizes the estimated CIF for the four methods in question along with the respective RSS values for both the exact and the empirical CIF's under q=1, which is the case of the NHPP.

Table 3 – Estimated CIF along with the respective RSS values for exact and empirical CIF's under q=1

t	n=10^7	n=400	PFF	Method 1	Method 2	Method 3	Method 4
0.050	0.003	0.003	0.003	0.007	0.004	0.003	0.003
0.100	0.010	0.013	0.013	0.018	0.013	0.011	0.011
0.150	0.022	0.028	0.028	0.034	0.028	0.024	0.025
0.200	0.040	0.040	0.038	0.054	0.048	0.043	0.044
0.250	0.063	0.060	0.058	0.079	0.073	0.066	0.068
0.300	0.090	0.100	0.093	0.109	0.103	0.095	0.098
0.350	0.122	0.155	0.138	0.143	0.138	0.128	0.133
0.400	0.160	0.183	0.155	0.181	0.177	0.166	0.172
0.450	0.202	0.230	0.193	0.225	0.221	0.209	0.217
0.500	0.250	0.273	0.228	0.273	0.270	0.257	0.267
0.550	0.302	0.328	0.268	0.325	0.323	0.310	0.321
0.600	0.360	0.370	0.298	0.382	0.381	0.367	0.380
0.650	0.422	0.443	0.350	0.444	0.443	0.429	0.443
0.700	0.489	0.525	0.408	0.510	0.509	0.496	0.510
0.750	0.562	0.583	0.440	0.580	0.580	0.567	0.582
0.800	0.639	0.630	0.463	0.656	0.655	0.644	0.658
0.850	0.721	0.720	0.505	0.736	0.735	0.725	0.738
0.900	0.809	0.835	0.555	0.820	0.819	0.810	0.822
0.950	0.901	0.915	0.590	0.909	0.907	0.901	0.909
1.000	0.998	1.003	0.613	1.003	0.999	0.996	1.001
RSS (relative to n=400)	n/a			0.002	0.002	0.004	0.003
RSS (relative to n=10^7)	0.007			0.006	0.004	0.001	0.004

Method 1 yielded the following Pade approximation: $W(t)$ = 0.088 t + 0.915t^2. The first term is small relative to the second one, while the second term is close to the exact CIF with the underlying Weibull distribution, which in case of $\hat{q}$=1 is $W(t)$ = 1.000 t^2. This is to say that the obtained Pade function approximates the exact solution reasonably well in this case.

Method 2 resulted in these estimates of the g–renewal parameters: $\hat{q}$=1.00, $\hat{\lambda}$=1.00, $\hat{\alpha}$=1.888, ε=0.046. Method 3 resulted in the following estimates: $\hat{q}$=1.00, $\hat{\lambda}$=0.999, $\hat{\alpha}$=1.955, ε=0.018. Obviously, this is the best approximation for renewal process parameters. Method 4 resulted in the following estimates: $\hat{q}$=0.720, $\hat{\lambda}$=1.043, $\hat{\alpha}$=1.993, ε=0.116.

All four methods performed reasonably well, but again, the winner is Method 3, if we compare the corresponding data with the exact solution in this interval and the error in the g–renewal parameters estimation.

Table 4 shows the summary of parameter estimation across all 4 methods. Notably, the relative error of Method 3 is within the range of the relative standard error (0.05) associated with the empirical data set (n=400), which was used for parameter estimation in all three cases of q. Other methods do not show the same consistent accuracy.

Table 4 – Summary of parameter estimates under various values of the restoration factor

	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{q}$	ε
Exact	**1.000**	**2.000**	**0.000**	**0.000**
Method 1		$-0.259\ t^{0.6}+1.023\ t^{1.2}$		
Method 2	0.949	1.675	0.020	0.147
Method 3	1.037	2.006	0.000	0.017
Method 4	1.009	1.998	0.000	0.004
Exact	**1.000**	**2.000**	**0.500**	**0.000**
Method 1		$5.581\text{-}11.147e^{-0.5t}+5.566e^{-t}$		
Method 2	1.054	1.964	0.140	0.160
Method 3	1.010	2.037	0.440	0.031
Method 4	1.070	2.105	0.160	0.158
Method 4+	1.028	2.055	0.500	0.027
Exact	**1.000**	**2.000**	**1.000**	**0.000**
Method 1		$0.088t + 0.915t^2$		
Method 2	1.000	1.888	1.000	0.046
Method 3	0.999	1.955	1.000	0.018
Method 4	1.043	1.993	0.720	0.116
Method 4+	1.040	1.988	0.780	0.091

3.3. Comparative Analysis of Extrapolation Accuracy

The ultimate measure of the effectiveness of an approximation is, of course, the extrapolation accuracy. It is important because most of the g–renewal applications involve the prediction of the expected (future) number of failures for the purpose of risk assessment, maintenance scheduling, spare logistics, etc. Shown in Figures 3–5 are extrapolations of the g–renewal functions corresponding to each of the four discussed methods under the three values of the g–renewal restoration parameter (q=0, 0.5, 1). Note that Method 4+ uses an improved approximation algorithm, which is discussed in the end of this section.

The worst is Method 1, except for the case q=1, when the exact solution is available in the form, which is included in the set of approximation functions utilized by this method. Unlike the other three methods, Method 1 does not use the additional information that the solution is the g–renewal CIF with underlying Weibull distribution function.

It is remarkable that in the case of q=0, all methods except the first one provided good extrapolation results. Even though the obtained renewal parameters in Method 2 are not close to the exact values, the corresponding CIF is close to the exact solution over a relatively large time interval. If q=0, the solution is not much sensitive to an error in the Weibull shape parameter estimation. For large time values, the following asymptotic formula is valid: $W(t) \approx t/\eta$, where $\eta = \Gamma(1+1/\alpha)/\lambda$ is the Weibull MTTF. For the Weibull parameters obtained in Method 2, $\eta = 0.942$. The exact value is $\eta = 0.886$. The difference is relatively small even though the shape parameter obtained in this method (1.675) is significantly less than the exact value (2.00).

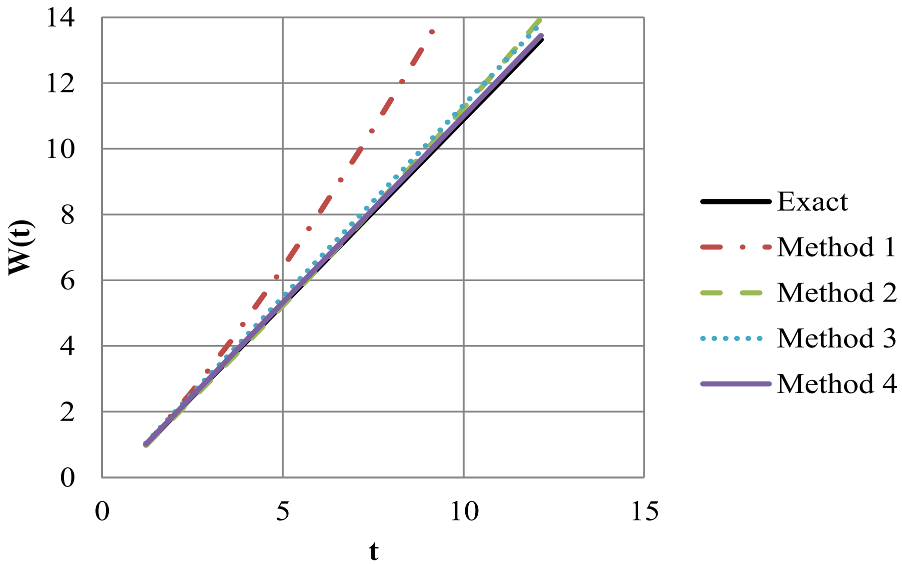

Figure 3 – Extrapolated CIF's corresponding to q=0 (see Table 1)

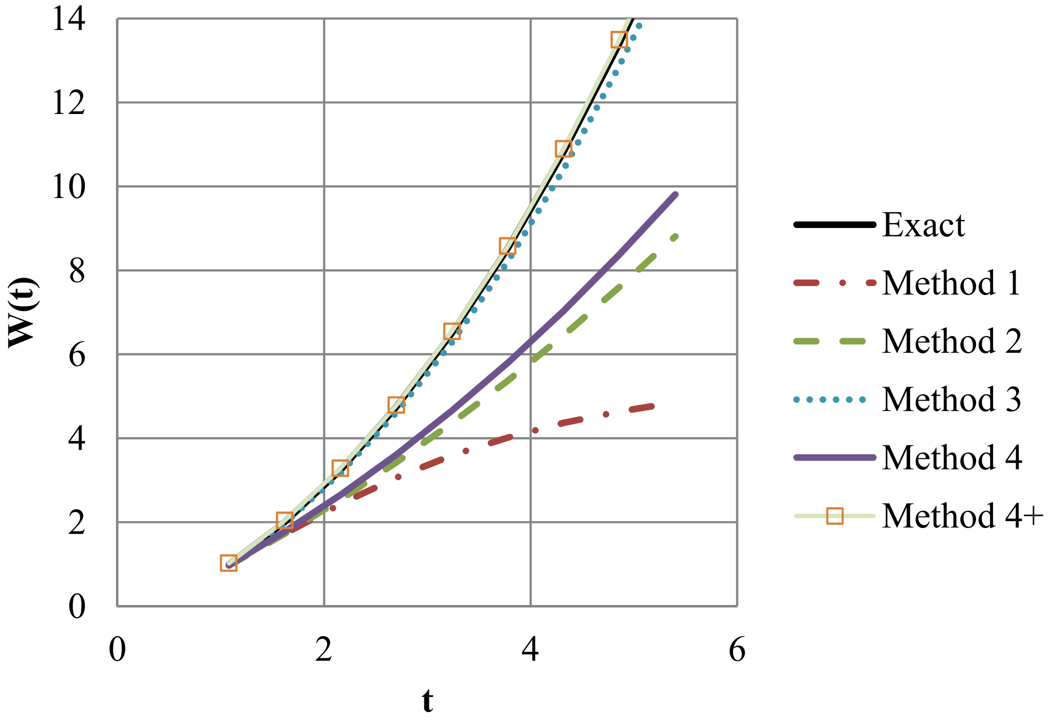

Figure 4 – Extrapolated CIF's corresponding to q=0.5 (see Table 2)

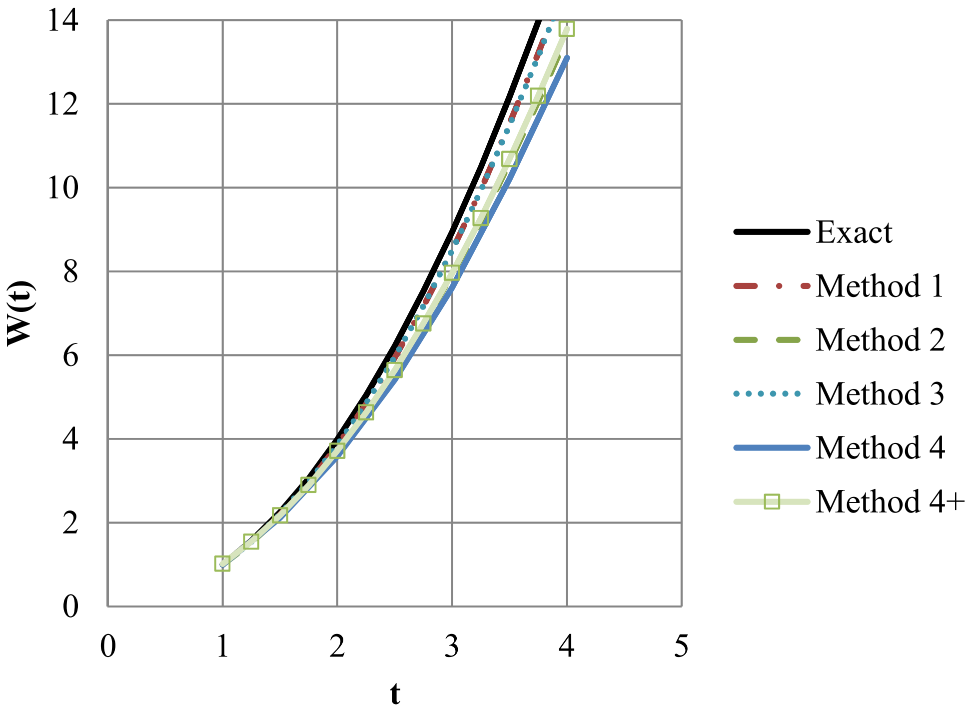

Figure 5 – Extrapolated CIF's corresponding to q=1 (see Table 3)

As expected, Method 3 is the best extrapolation method in all considered cases. It has the following advantages:

1. It is a stable, regularized method for solving the inverse problem of g–renewal CIF estimation and the subsequent extrapolation.
2. In this method, the underlying distribution parameters are estimated separately from the restoration factor. This subtask is very well developed and any suitable method (e.g., MLE) can be applied efficiently. Moreover, the type of the distribution can be selected properly at this stage according to experimental data: not only Weibull distribution can be considered.
3. The volume of statistical data is increased essentially in this method compared to other 3 methods. This leads to increase of extrapolation accuracy.
4. Method 3 is much simpler for calculation because, in its second step (solving the g–renewal equa-

tions) only one (restoration factor) parameter is estimated. Even a Monte Carlo simulation method can be selected in this case as the estimation method.

Method 4 is the second best to be recommended, especially if the data are limited by the number of observed repairable systems. This method yields good extrapolation results for q=0, because in this case the results do not depend much on accuracy of Weibull parameters estimation. It is also good if q=1, because Eq. (5) applied for Weibull parameters estimation is exact. However, for q=0.5 Method 4 failed because (5) is limited by a small time interval. Here we suggest a better approximation for CIF.

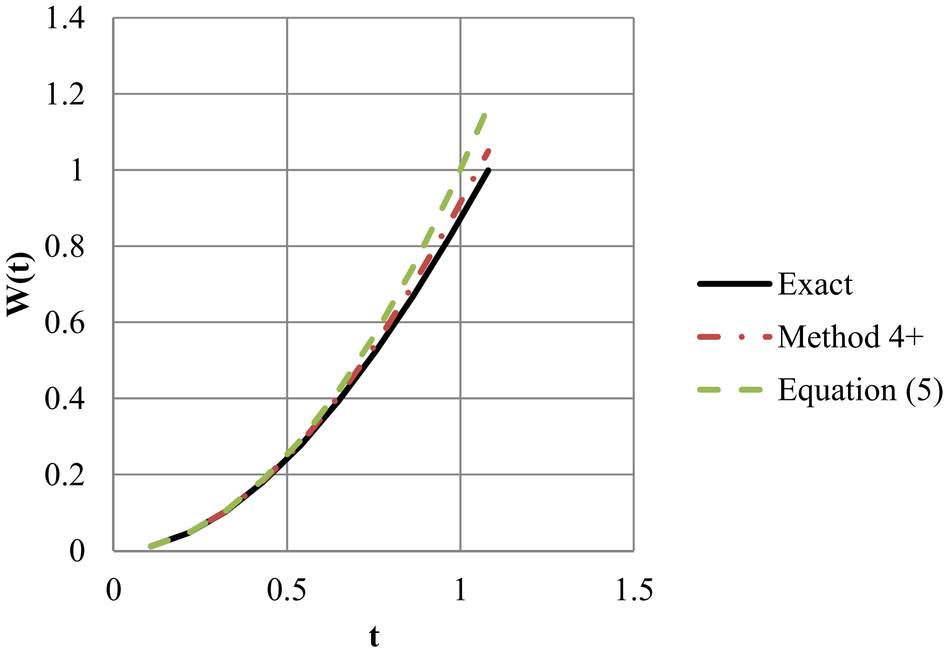

Figure 6 – Improved approximation (Method 4+) of the g–renewal function for q=0.5.

Equation (5) yields upper bound estimation for the g–renewal function if Weibull shape parameter is greater than 1. The lower bound estimation can be derived from [16] in the following form:

$$W(t) \approx F(t) + I(t) \tag{7}$$

where

$$I(t) = \int_0^t F_q dF(x),$$

$$F_q(t,x) = 1 - \exp\left(-\lambda^{\alpha}\left[\left(t-(1-q)x\right)^{\alpha} - (qx)^{\alpha}\right)\right]\right)$$

We consider the approximation for the g–renewal function as the average value of (5) and (7), which is represented in the Fig. 6 along with exact solution and approximation (5) for the case when q=0.5. We significantly expanded the time interval when the approximation is valid.

We constructed the following iteration algorithm:

$$(\lambda_i t)^{\alpha_i} = W + \Delta_{i-1} \tag{8}$$

where Δ_{i-1} is defined in the previous step of itereation as the follows:

$$\Delta_{i-1} = 0.5\left(\left(\lambda_{i-1}t\right)^{\alpha_{i-1}} - F_{i-1} - I_{i-1}\right)$$

and $\Delta_0 = 0$. The latter equation implies that at the very first step we use approximation (5), then this result is improving during the iteration.

Using the described iteration algorithm, we obtained the following estimates of the g–renewal parameters for the case of q=0.5: $\hat{q}$=0.44, $\hat{\lambda}$=1.028, $\hat{\alpha}$=2.055 (c.f., previously obtained results $\hat{q}$ = 0.16, $\hat{\lambda}$=1.07, $\hat{\alpha}$=2.105). It is remarkable that in case of q=1, we also obtained a better solution $\hat{q}$=0.78, λ=1.040, $\hat{\alpha}$=1.988 (c.f., previously obtained results $\hat{q}$=0.720, $\hat{\lambda}$=1.043, $\hat{\alpha}$=1.993), because in our approximation of empirical data $q \neq 1$ and formula (7) is more accurate than (5).

Denoted by “+” in Fig. 4 is the improved approximation of Method 4 under q=0.5. It is evident that the improved approximation is practically indistinguishable from the exact solution in this case. Denoted by “+” in Fig. 5 is the improved approximation under q=1, which makes it considerably better than the original Method 4 and also better that Method 2.

4. A case study

Consider the cumulative operational time at failures on 6 repairable systems discussed by Mettas and Zhao [3] in Table 5.

Table 5 – Cumulative failure arrival times on 6 repairable systems [3].

	System 1	System 2	System 3	System 4	System 5	System 6
Start	0	0	0	0	0	0
End	8760	5000	6200	1300	2650	500
1	2227	773	901	411	689	106
2	2733	1034	1290	1123	915	
3	3524	3011	2690			
4	5569	3121	3929			
5	5886	3624	4328			
6	5946	3758	4704			
7	6018		5053			
8	7203		5473			

Shown in Table 6 are the ML estimates of the GRP parameters (with the underlying Weibull distribution) obtained by Mettas and Zhao [3] based on the data in Table 5 for various GRP models. Figure 5 shows the respective CIF's corresponding to GRP parameter estimates in Table 6.

Table 6 – MLE of GRP parameters based on Table 5 [3].

	ORP	NHPP	GRP (Kijima-I)	GRP (Kijima-II)
$\hat{\alpha}$	0.0004	0.001	0.00018	0.000068
$\hat{\lambda}$	1.1409	1.113	1.23863	1.358201
$\hat{q}$	0	1	0.10599	0.552159

Even though the estimation method used by Mettas and Zhao is different than ours, their results are yet another illustration of the conjecture we have proposed: the problem of estimating parameters of a g–renewal process is a mathematically ill–posed problem. The estimated CIF's in Figure 6 are practically indistinguishable relative to each other despite the vast difference in probabilistic models and statistical estimates of those models' parameters shown in Table 6.

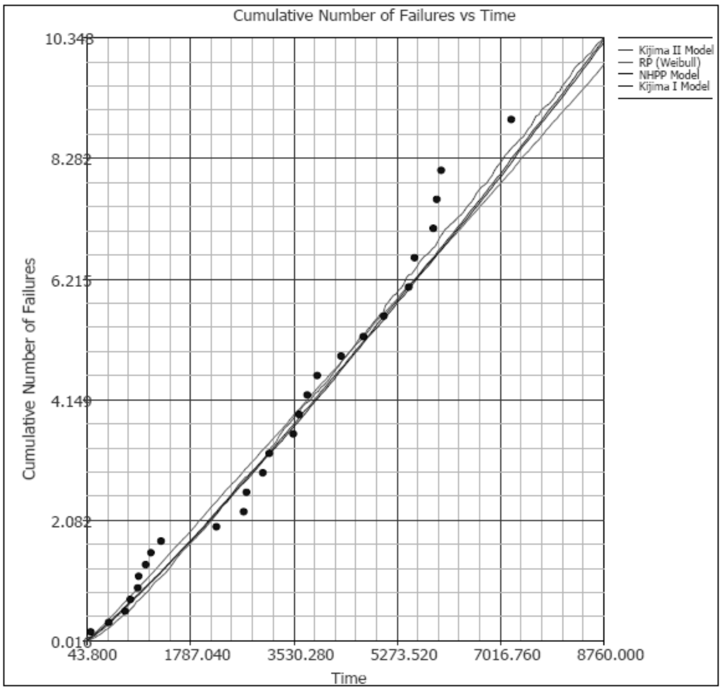

Figure 7 – CIF's corresponding to GRP parameter estimates in Table 6 [3]

To demonstrate the efficiency of our methods, we have used the first 10 points of the Mettas & Zhao non-parametric CIF (see Fig. 6) to estimate the parameters of the g–renewal function, obtain a corresponding extrapolation and then compare it with the actual non-parametric estimate at the respective times. Note that unlike Mettas & Zhao, whose task was to parametrically fit the data and utilize the obtained model for *interpolation*, we are solving a much more challenging problem: use the initial 20% of the data to estimate the model, and the remaining 80% – to validate the model's *extrapolation*.

Shown in Table 7 are the 10 points of nonparametric estimates of the empirical CIF and the percent of the first failures corresponding to the data in Table 5. Again, we will use the PFF only in Method 4.

Table 7 – Nonparametric estimates of the empirical CIF and the percent of first failures corresponding to the first 10 data points in Fig 7

t	106	411	689	733	901	915	1034	1123	1290	2227
CIF	0.167	0.333	0.533	0.733	0.933	1.133	1.333	1.533	1.733	1.983
NFF	0.167	0.333	0.500	0.667	0.833	–	–	–	–	–

Table 8 summarizes parameter estimates of the g–renewal function obtained by the 3 discussed methods. It is interesting to note that in all methods, we obtained the estimates of Weibull shape parameter slightly less than 1. Strictly speaking, it is in agreement with the first 10 data points in Figure 6 -- corresponding piece of the curve concaves up in 0.4 < *W(t)* < 2.1.

Table 8 – GRP parameter estimates based on the data in Table 7

Method	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{q}$
Method 2	0.00115	0.8403	1.0
Method 3	0.00134	0.9492	1.0
Method 4	0.00108	0.9233	0.8

Figure 8 shows the CIF extrapolations based on the GRP parameters in Table 8. Even though in the considered case study we restricted amount of data (10 points corresponding to only 6 systems), Method 3 performed quite well in the *entire* time interval. Method 2 works well in the interval of *W(t)* < 4. Method 4 appears better than Method 2; it works in the interval of *W(t)* < 5. Note that *q* is close to 1 in this case, so approximation (5) works reasonably well. We were unable to apply Method 4+ in this case, because it works only if Weibull shape parameter α ≥1.0.

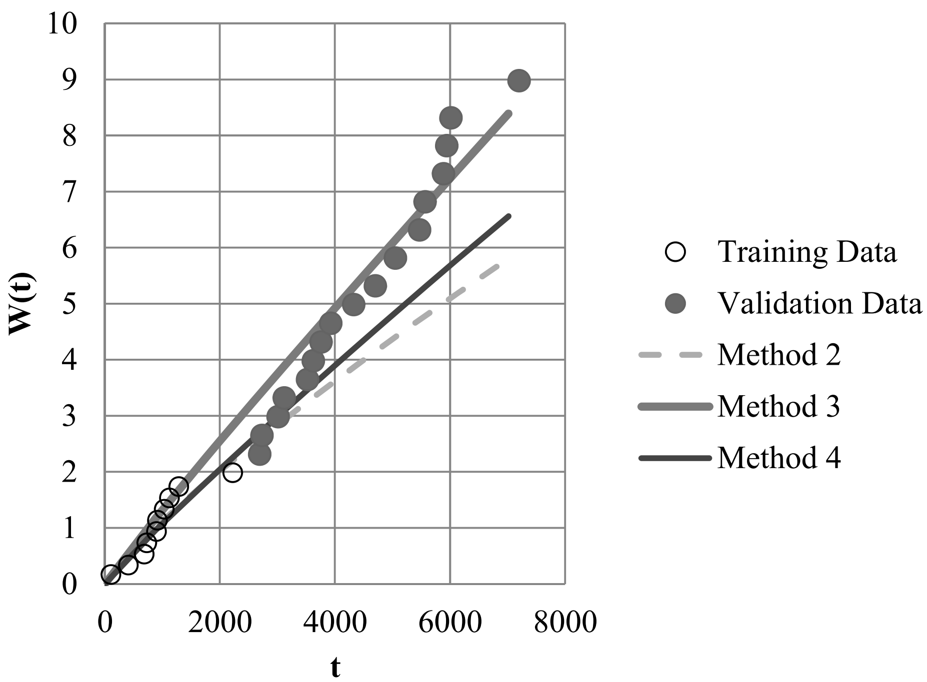

Figure 8 – CIF extrapolation based on the GRP parameters in Table 8

5. Conclusions

We have shown that the extrapolation of the CIF of the g–renewal process is a mathematically incorrect inverse problem, in general case. For regularization of this problem, special methods and/or additional information is required. We suggested a regularization approach of separating the estimation of the underlying TTF distribution from the estimation of the g–renewal restoration factor. The approach is based on the use of the TFF information, which is contained (explicitly or implicitly) in the g–renewal process data. The conducted simulation studies as well as the considered practical case study confirmed the improved extrapolation accuracy of the proposed methods.

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Аннотация

СТАТИСТИЧЕСКОЕ ОЦЕНИВАНИЕ ПРОЦЕССА G-ВОССТАНОВЛЕНИЯ КАК ПЛОХО ОБУСЛОВЛЕННАЯ ОБРАТНАЯ ЗАДАЧА

Кривцов В. В., Евкин А. М.

Статистическое оценивание процесса g-восстановления рассматривалось многими исследователями. В данной статье мы показываем, что эта задача относится к классу плохо обусловленных, обратных задач (для которых решение неуникально и/или очень чувствительно к статистическому разбросу данных) и требует регуляризации по Тихонову. Вне зависимости от метода оценивания, существующие методы обычно включают в соответствующую целевую функцию параметры базового распределения одновременно с параметром восстановления. Мы же предлагаем регуляризировать эту задачу путем раздельного последовательного оценивания вышеупомянутых параметров. Используя симуляционное исследование, мы показываем, что экстраполяционная точность предложенного метода гораздо выше, чем у существующих.

Анотація

СТАТИСТИЧНЕ ОЦІНЮВАННЯ ПРОЦЕСУ G–ВІДНОВЛЕННЯ ЯК ПОГАНО ОБУМОВЛЕНА ОБЕРНЕНА ЗАДАЧА

Крівцов В. В., Євкін А. М.

Статистичне оцінювання процесу g–відновлення розглядалося багатьма дослідниками. У цій статті ми показуємо, що ця задача відноситься до класу погано обумовлених, обернених задач (для яких рішення не унікальне і/або дуже чутливо до статистичного розкиду даних) і вимагає регулярізації по Тихонову. Незалежно від методу оцінювання, існуючі методи зазвичай включають у відповідну цільову функцію параметри базового розподілу одночасно з параметром відновлення. Ми ж пропонуємо регуляризувати це завдання шляхом роздільного послідовного оцінювання вищезгаданих параметрів. Використовуючи дослідження симуляції, ми показуємо, що екстраполяційна точність запропонованого методу набагато вища, ніж у існуючих.