MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

STATE BIOTECHNOLOGICAL UNIVERSITY

Faculty of Mechatronics and Engineering

Department of Physics and Mathematics

LINEAR ALGEBRA

Guidelines for Students Studying a Course of Mathematics in English

for the first (bachelor) level of Higher Education
Specialty 281 "Public Management and Administration"

Kharkiv
2023

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Guidelines cover matrices, determinants and systems of linear equations. Solutions of typical problems are given. Guidelines contain tasks for independent work. It is recommended for students studying mathematics in English.

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1. Matrices

1.1. Basic Definitions

Definition. A rectangular array of numbers is called a ***matrix***.

We denote matrices by Latin letters *A, B, C,* ....

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

This matrix has *m* rows and *n* columns. We call *A* a “*m* by *n*” matrix or a matrix of [*m* x *n*] dimension.

The element in the *i*-th row and *j*-th column of a matrix can be represented by a_{ij}. Matrix *A* can be represented as $A_{m\times n} = (a_{ij})$. Matrix can also be enclosed in brackets $A = [a_{ij}]$ or braces $A = \{a_{ij}\}$.

Definition. Two matrices *A* and *B* are ***equal*** if and only if they have the same elements in the same positions.

$$A = B \;\Leftrightarrow\; (a_{ij} = b_{ij}).$$

For example,

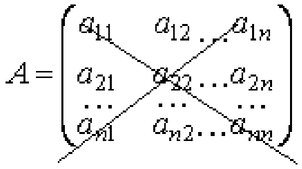
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},\; B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},\; A = B.$$

Let's consider ***forms of matrices***.

Matrices $A_{m\times 1}=\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_m \end{pmatrix}$ and $B_{1\times n}=(b_1\ b_2\ \dots b_n)$ are ***vector-matrices***. A matrix $B_{1\times n}$ is called a ***row vector***. A matrix $A_{m\times 1}$ is called a ***column vector***.

If $m=n$, then a matrix is called a ***square matrix***. Its order is equal *n*.

Let the square matrix A be given. The diagonal containing $a_{11}, a_{22}, \dots, a_{n-1n-1}, a_{nn}$ is called the ***principal (main) diagonal***. Another diagonal is called the ***secondary (minor) diagonal***.



If there are *nonzero* elements on the main diagonal of a square matrix *A*, then this matrix is called a ***diagonal matrix***. For example, *A* is diagonal matrix with order 3

$$A=\begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

4. A diagonal matrix is said to be a ***unit matrix*** if all diagonal elements equal 1. It is denoted by *E*. For example, unit matrix with order 4 is

$$E=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Matrix all non-zero elements of which situated under (over) its main diagonal is called a ***triangular matrix*.** There are two following examples of the triangular matrices

$$\begin{pmatrix} 2 & -7 & 0 & 1 \\ 0 & 6 & 9 & -4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{pmatrix}.$$

If all elements of a matrix are equal to zero then matrix is called a ***zero matrix***. Denote as O

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now consider operations with matrices.

1.2 Operations with Matrices

(A) Transposition of a Matrix

If we interchange columns and rows of a matrix *A*, we get the ***transposed matrix*** A^T.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}.$$

For example,

$$A = \begin{pmatrix} 2 & 9 \\ 6 & 1 \\ 3 & 0 \end{pmatrix}, \quad A^T = \begin{pmatrix} 2 & 6 & 3 \\ 9 & 1 & 0 \end{pmatrix}.$$

There are properties of transposed matrices

1) $\left(A^T\right)^T = A$;

2) $(\lambda A)^T = \lambda A^T$;

3) $(A+B)^T = A^T + B^T$;

4) $(A\cdot B)^T = B^T \cdot A^T$.

Here A and B are matrices, λ is a number.

(B) Addition (Subtraction) of Matrices

We can add and subtract the matrices of the ***same dimension***. Their sum (difference) is the matrix we get by adding (subtracting) corresponding elements in the given matrices:

$$C = A \pm B \quad \Leftrightarrow \quad c_{ij} = a_{ij} \pm b_{ij},\ \forall i = \overline{1,m},\ \forall j = \overline{1,n}.$$

Example. Let the matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ 4 & 7 \end{pmatrix}$ be given. Find their sum $A+B$ and difference $A-B$.

Solution

$$A+B = \begin{pmatrix} 1+2 & 2+3 \\ 3+0 & 4+1 \\ 5+4 & 6+7 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 3 & 5 \\ 9 & 13 \end{pmatrix},\quad A-B = \begin{pmatrix} 1-2 & 2-3 \\ 3-0 & 4-1 \\ 5-4 & 6-7 \end{pmatrix} = \begin{pmatrix} -1 & -15 \\ 3 & 3 \\ 1 & -1 \end{pmatrix}.$$

(C) Scalar Multiplication

To multiply a matrix A by a number λ, we multiply each element of this matrix by λ

$$A = (a_{ij}) \Leftrightarrow \lambda A = (\lambda a_{ij}).$$

For example, $A = \begin{pmatrix} 3 & 5 \\ \frac{7}{4} & -2 \end{pmatrix}$. Then $4A = \begin{pmatrix} 12 & 20 \\ 7 & -8 \end{pmatrix}$.

Addition of matrices and scalar multiplication are called *linear operations*. Linear operations with matrices have the ***properties***.

1) $A+B=B+A$;

2) $A+(B+C)=(A+B)+C$;

3) $A+O=A$;

4) $A+(-A)=O$;

5) $(\alpha+\beta)A=\alpha A+\beta A$;

6) $1\cdot A=A$;

7) $\alpha(A+B)=\alpha A+\alpha B$;

8) $\alpha(\beta A)=(\alpha\beta)A$.

Here A, B and C are matrices, α and β are real numbers, O is a zero matrix.

(D) Multiplication of Matrices

If the number of columns of a matrix A equals the number of rows of a matrix B, then the product $A\cdot B$ is defined. The multiplication a matrix $A_{m\times k}$ by a matrix $B_{k\times n}$ is a matrix $C_{m\times n}$ whose element c_{ij} is the product of the i-th row of A and the j-th column of B

$$A_{m\times \underline{k}}\cdot B_{\underline{k}\times n}=C_{m\times n},\; c_{ij}=\sum_{l=1}^{k}a_{il}\cdot b_{lj}\,.$$

In general

$$A\cdot B\neq B\cdot A.$$

Multiplication of matrices has the following ***properties***:

1) $A(BC)=(AB)C$;

2) $\lambda(AB)=(\lambda A)B=A(\lambda B)$;

3) $C(A+B)=CA+CB$;

4) $(A+B)C = AC + BC$;

5) $A\cdot E = E\cdot A = A$;

6) $A\cdot O = O\cdot A = O$.

Here A, B and C are matrices, λ is a real number, E is a unit matrix O is a zero matrix.

Example.

Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 3 \\ 1 & 5 \\ -1 & 1 \end{pmatrix}$. Find $C = AB$, $D = BA$.

Solution.

$$C_{2\times 2} = A_{2\times 3}\cdot B_{3\times 2};$$

$$C = A\cdot B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix}\cdot\begin{pmatrix} 0 & 3 \\ 1 & 5 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2\cdot 0+1\cdot 1+1\cdot(-1) & 2\cdot 3+1\cdot 5+1\cdot 1 \\ 0\cdot 0+3\cdot 1+2\cdot(-1) & 0\cdot 3+3\cdot 5+2\cdot 1 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 1 & 17 \end{pmatrix}.$$

$$D_{3\times 3} = B_{3\times 2}\cdot A_{2\times 3};$$

$$D = B\cdot A = \begin{pmatrix} 0 & 3 \\ 1 & 5 \\ -1 & 1 \end{pmatrix}\cdot\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 0\cdot 2+3\cdot 0 & 0\cdot 1+3\cdot 3 & 0\cdot 1+3\cdot 2 \\ 1\cdot 2+5\cdot 0 & 1\cdot 1+5\cdot 3 & 1\cdot 1+5\cdot 2 \\ -1\cdot 2+1\cdot 0 & -1\cdot 1+1\cdot 3 & -1\cdot 1+1\cdot 2 \end{pmatrix} = \begin{pmatrix} 0 & 9 & 6 \\ 2 & 16 & 11 \\ -2 & 2 & 1 \end{pmatrix}.$$

As we see in this case $AB \neq BA$.

2 Determinants

2.1 Basic Definitions

With a square matrix A we associate a number called the ***determinant***.

Definition. ***Determinant of a square matrix*** A is called a number that calculated by the certain rule.

We denote determinant as Δ, $|A|$ or det *A*. The determinant of the *n*-th order for a square matrix A of the n-th order is of the form

$$\Delta = \det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

For $n=1$, $A=(a_{11})$ we have

$$\det A = a_{11}.$$

For $n=2$, $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ we have the definition

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}.$$

Example.

$$\Delta = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = -2.$$

The determinant of the third order can be calculated by the formula

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -$$

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

Definition. A ***minor*** M_{ij} of an element a_{ij} of the determinant with order *n* is the determinant with order $n-1$ obtained from the given determinant by deleting the ***i***-th row and ***j***-th column.

Definition. The quantity $A_{ij} = (-1)^{i+j} M_{ij}$ is called a ***cofactor*** of an element a_{ij}.

Let's consider the ***example***. Calculate M_{23} and A_{23} of the determinant

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 5 & 3 \\ 7 & 3 & 5 \end{vmatrix}.$$

We have

$$M_{23} = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 5 & 3 \\ 7 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 7 & 3 \end{vmatrix} = 3 - 14 = -11.$$

$$A_{23} = (-1)^{2+3} M_{23} = -(-11) = 11.$$

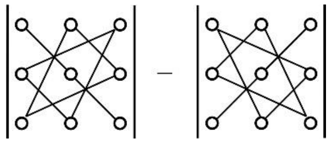
In general case we such definition of a determinant.

Definition. ***Determinant of a square matrix*** A of the n-th order is called a number function $\det A$ that calculated by the formula

$$\det A = a_{11}M_{11} - a_{21}M_{21} + \ldots + (-1)^{i+1} a_{i1}M_{i1} + \ldots + (-1)^{n+1} a_{n1}M_{n1} =$$
$$= \sum_{i=1}^{n} (-1)^{i+1} a_{i1}M_{i1},$$

where M_{i1} is the minor of an element a_{ij}.

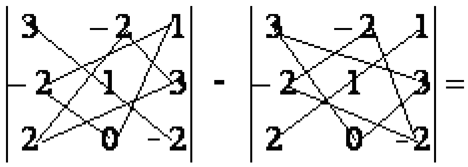
It follows that we can calculate determinants of the third order using the ***rule of triangles***. We replace elements by dots and construct main and secondary diagonal as well as triangles

$\det A = +$ 

Example. Let's show calculation of a determinant of third order

$$\Delta = \begin{vmatrix} 3 & -2 & 1 \\ -2 & 1 & 3 \\ 2 & 0 & -2 \end{vmatrix},$$

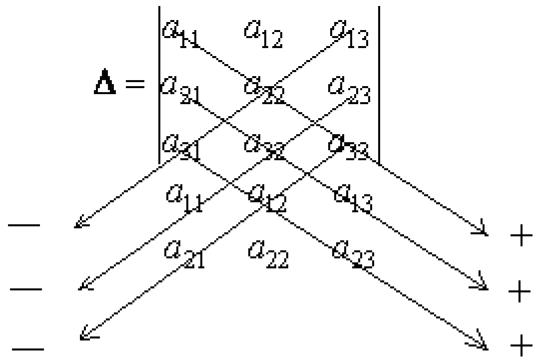
Solution.

$\Delta = +$ 

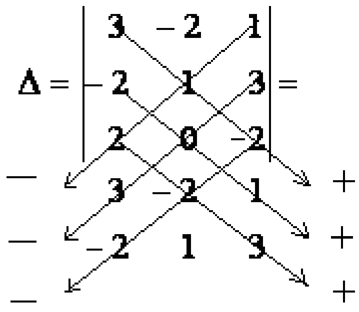
$$= 3\cdot 1\cdot(-2) + 2\cdot(-2)\cdot 3 + 1\cdot(-2)\cdot 0 - 1\cdot 1\cdot 2 - (-2)\cdot(-2)\cdot(-2) - 3\cdot 3\cdot 0 =$$

$$= -6 - 12 + 0 - 2 + 8 - 0 = -20 + 8 = -12\,.$$

The determinant of the third order can be calculated also by the ***rule of addition of rows or columns*****.** Write a determinant of the third order. Add the first and the second rows. Construct the main diagonal and its parallels; take sum of products of elements located on main diagonal and its parallels. Construct the secondary diagonal and its parallels; subtract products of elements located on the secondary diagonal and its parallels.



As an ***example***, consider the same determinant using the rule.



$$= 3 \cdot 1 \cdot (-2) + 2 \cdot (-2) \cdot 3 + 1 \cdot (-2) \cdot 0 - 1 \cdot 1 \cdot 2 - (-2) \cdot (-2) \cdot (-2) - 3 \cdot 3 \cdot 0 =$$

$$= -6 - 12 + 0 - 2 + 8 - 0 = -20 + 8 = -12 \,.$$

We have obtained the same result.

2.2 Basic Properties of Determinants

Now we state the basic properties of determinants.

1) The determinant of the transposed matrix A^T is equal to the given determinant A.

$$\det A = \det A^T \,.$$

2) If two rows (columns) of a determinant are interchanged, the determinant changes its sign.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \Delta' = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix}, \quad \Delta = -\Delta'.$$

3) The common multiplier of the row (column) can be taking out before the determinant sign.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \lambda \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Here λ is a real number.

4) A determinant is equal to zero, if the elements of the row (column) are equal to zero.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ 0 & 0 \end{vmatrix} = 0.$$

5) If two rows (columns) of a determinant coincide (or are proportional), the determinant is zero.

6) A determinant does not change if we add to all elements of a row (column) of the determinant the corresponding elements of other row (column) multiplied by some number.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{11} + a_{21} & \lambda a_{12} + a_{22} & \lambda a_{13} + a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

7) If each element of some row (column) of a determinant represents the sum of two elements, the equality takes place

$$\begin{vmatrix} a'_{11} + a''_{11} & a_{12} & a_{13} \\ a'_{21} + a''_{21} & a_{22} & a_{23} \\ a'_{31} + a''_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a'_{11} & a_{21} & a_{31} \\ a'_{12} & a_{22} & a_{32} \\ a'_{13} & a_{23} & a_{33} \end{vmatrix} + \begin{vmatrix} a''_{11} & a_{12} & a_{13} \\ a''_{21} & a_{22} & a_{23} \\ a''_{31} & a_{32} & a_{33} \end{vmatrix}.$$

8) The determinant of a triangle or diagonal matrix is equal to the product of the elements of the main diagonal.

Examples.

a) $\Delta = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 1\cdot 2 - 1\cdot 2 = 0$.

b) $\Delta = \begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 3\cdot 4 - 6\cdot 2 = 0$.

c) $\Delta = \begin{vmatrix} 2 & 0 & 0 \\ 5 & -2 & 0 \\ 1 & 4 & 7 \end{vmatrix} = 2\cdot(-2)\cdot 7 = -28.$

Consider the theorem concerning calculation of determinants.

Theorem (*Laplace's Theorem*). A determinant is equal to the sum of products of elements of any row (column) on their algebraic cofactors

$$\Delta = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in},\ \forall i = \overline{1, n}.$$

For a determinant of the third order we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}.$$

This is expanding of a determinant by the first row.

Consequence. The sum of products of elements of any row (column) on cofactors of elements of other row (column) with corresponding numbers is equal to zero

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \ldots + a_{in}A_{kn} = 0,\ \forall i = \overline{1, n} \wedge k \neq i.$$

For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} = 0.$$

Examples.

a) Calculate determinant of the fourth order by using Laplace's Theorem

$$\Delta = \begin{vmatrix} 1 & 0 & 2 & 3 \\ 2 & 0 & 2 & 0 \\ 3 & 1 & 4 & 5 \\ 5 & 0 & 0 & 0 \end{vmatrix}.$$

***Solution*.**

A row or column with many zeroes suggests a Laplace's expansion. We can compute the determinant by expanding along the fourth row

$$\Delta = 5 \cdot (-1)^{4+1} \begin{vmatrix} 0 & 2 & 3 \\ 0 & 2 & 0 \\ 1 & 4 & 5 \end{vmatrix} + 0 + 0 + 0 = (-5) \cdot 1 \cdot (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} = -5 \cdot (2 \cdot 0 - 2 \cdot 3) = 30.$$

b) Calculate determinant of the fourth order by using properties of determinants

$$\Delta = \begin{vmatrix} 5 & 1 & 3 & 5 \\ -4 & 2 & 1 & 3 \\ 4 & 0 & 2 & 1 \\ 2 & 4 & 7 & 1 \end{vmatrix}.$$

***Solution*.**

Let's multiply the first row by (–2) and add to the second row, multiply the first row by (–4) and add to the fourth row, and expand the determinant by the second column.

$$\Delta = \begin{vmatrix} 5 & 1 & 3 & 5 \\ -4 & 2 & 1 & 3 \\ 4 & 0 & 2 & 1 \\ 2 & 4 & 7 & 1 \end{vmatrix} \cdot (-2) \quad \cdot (-4) = \begin{vmatrix} 5 & 1 & 3 & 5 \\ -14 & 0 & -5 & -7 \\ 4 & 0 & 2 & 1 \\ -18 & 0 & -5 & -19 \end{vmatrix} =$$

$$= 1 \cdot (-1)^{1+2} \begin{vmatrix} -14 & -5 & -7 \\ 4 & 2 & 1 \\ -18 & -5 & -19 \end{vmatrix} = \begin{vmatrix} 14 & 5 & 7 \\ 4 & 2 & 1 \\ -18 & -5 & -19 \end{vmatrix}.$$

Multiply the second row by (–7) and add to the first row, multiply the second row by 19 and add to the third row. Expand the determinant by the third column.

$$\Delta = \begin{vmatrix} -14 & -9 & 0 \\ 4 & 2 & 1 \\ 58 & 33 & 0 \end{vmatrix} = (-1)^{2+3} \cdot \begin{vmatrix} -14 & -9 \\ 58 & 33 \end{vmatrix} = \begin{vmatrix} 14 & 9 \\ 58 & 33 \end{vmatrix} = -60.$$

2.3 The Inverse of a Square Matrix

Definition. If the determinant of a square matrix A is equal to zero, then matrix A is called a ***singular*** matrix.

If the determinant of a square matrix A is not equal to zero, then matrix A is called a ***non-singular*** matrix.

Definition. A square matrix A^{-1} is called an ***inverse matrix*** of a matrix A if

$$A \cdot A^{-1} = A^{-1} \cdot A = E,$$

where E is a unit matrix.

Definition. Transposed matrix $\tilde{A}$ of cofactors of the corresponding elements of the given matrix A is called the ***adjoint matrix*** of A

$$\tilde{A} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}.$$

Theorem. A matrix A has an inverse matrix A^{-1} if and only if its determinant is not equal to zero.

So, an inverse matrix A^{-1} is calculated by the formula

$$A^{-1} = \frac{1}{\det A} \cdot \tilde{A} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}. \qquad (1)$$

Example. Find A^{-1} for

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Solution.

$$\det A = \begin{vmatrix} 2 & 1 & -1 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -1 \\ -1 & 3 & 0 \\ 5 & 2 & 0 \end{vmatrix} = -1\cdot(-1)^{1+3}\begin{vmatrix} -1 & 3 \\ 5 & 2 \end{vmatrix} = 17 \neq 0.$$

Thus, A^{-1} exists. Find all cofactors A_{ij} of elements a_{ij} and calculate A^{-1}.

$$A_{11} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6, \quad A_{21} = -\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = -2, \quad A_{31} = \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 3,$$

$$A_{12} = -\begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} = 2, \quad A_{22} = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5, \quad A_{32} = -\begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} = 1,$$

$$A_{13} = \begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} = -3, \quad A_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 1, \quad A_{33} = \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} = 7,$$

$$A^{*} = \begin{pmatrix} 6 & -2 & 3 \\ 2 & 5 & 1 \\ -3 & 1 & 7 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} \frac{6}{17} & \frac{-2}{17} & \frac{3}{17} \\ \frac{2}{17} & \frac{5}{17} & \frac{1}{17} \\ \frac{-3}{17} & \frac{1}{17} & \frac{7}{17} \end{pmatrix}.$$

Verify that $A^{-1}\cdot A = E$.

$$A\times A^{-1} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \times \begin{pmatrix} \frac{6}{17} & \frac{-2}{17} & \frac{3}{17} \\ \frac{2}{17} & \frac{5}{17} & \frac{1}{17} \\ \frac{-3}{17} & \frac{1}{17} & \frac{7}{17} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3 Methods of Solution of Systems of Linear Algebraic Equations

3.1 Basic Definitions

Definition. The system of equation

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 \ldots + a_{nn}x_n = b_n, \end{cases} \quad (2)$$

is called the ***system of n linear algebraic equation with n unknowns***.

Let’s write the system (2) in the matrix form

$$AX = B. \quad (3)$$

Here A is the ***system matrix***, X is the ***column of unknowns***, B is the ***column of the constant terms***.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{pmatrix}.$$

Definition. A ***solution of linear system*** (3) is called a set of real numbers which being is substituted in system (3) instead of unknowns, transforms all equations of the system into identities.

Two systems are called ***equivalent***, if sets of their solution coincide. We consider as equivalent also systems which have no solutions.

Definition. A system (2) having at least one solution is called a ***compatible*** system and ***incompatible system***, if there are no solutions.

Definition. A system (2) is called a ***definite system***, if it has a unique solution and ***indefinite system***, if it has more than one solution.

3.2 The Matrix Method

Assume that the matrix A is *non-singular* matrix, $\det A \neq 0$. It means that A^{-1} exists. Multiply the left and right parts of the equation (1.3) by A^{-1}. In this case we have

$$A \cdot X = B \Rightarrow A^{-1}AX = A^{-1}B \Rightarrow \left(A^{-1}A\right)X = A^{-1}B \Rightarrow$$

$$\left| \ A^{-1}A = E, \ EX = X \ \right| \Rightarrow X = A^{-1}B.$$

Thus, the solution system (1.3) is of the following form

$$X = A^{-1}B. \qquad (4)$$

If $\det A \neq 0$, then the system (3) has the ***unique solution*** defined by the formula (4).

Example. Solve a system of equations by the matrix method

$$\begin{cases} 4x_1 + 3x_2 + 2x_3 = 33, \\ 3x_1 + 2x_2 + x_3 = 23, \\ x_1 + x_2 + 2x_3 = 12. \end{cases}$$

Solution.

The matrix form of the given system is

$$A \cdot X = B,$$

where

$$A = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \; B = \begin{pmatrix} 33 \\ 23 \\ 12 \end{pmatrix}.$$

Determinant of the system matrix is

$$\Delta = \begin{vmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -1 \neq 0.$$

Thus, we are able to use the formula (4). As we know

$$A^{-1} = \frac{1}{\Delta} \cdot \tilde{A}.$$

Let us find $\tilde{A}$

$$\tilde{A} = \begin{pmatrix} 3 & -4 & -1 \\ -5 & 6 & 2 \\ 1 & -1 & -1 \end{pmatrix}.$$

As $\Delta = -1$, we have

$$A^{-1} = \begin{pmatrix} -3 & 4 & 1 \\ 5 & -6 & -2 \\ -1 & 1 & 1 \end{pmatrix}.$$

Check up

$$A \cdot A^{-1} = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -3 & 4 & 1 \\ 5 & -6 & -2 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$X = A^{-1} \cdot B = \begin{pmatrix} -3 & 4 & 1 \\ 5 & -6 & -2 \\ -1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 33 \\ 23 \\ 12 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}.$$

Thus, the solution of the given system is $x_1 = 5$; $x_2 = 3$; $x_3 = 2$ or

$$X = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}.$$

3.3 The Cramer's Rule

Let's consider the system of n linear algebraic equations with n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases} \tag{5}$$

Let a matrix A is non-singular matrix, $\det A \neq 0$. Determinant of the system is

$$\det A = \Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Then this system has a unique solution that can be calculated by the **Cramer's formulas**

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta},\ \dots\ x_n = \frac{\Delta_n}{\Delta}, \tag{6}$$

where

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix},\ \Delta_2 = \begin{vmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & b_n & \dots & a_{nn} \end{vmatrix}, \dots \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & b_n \end{vmatrix}.$$

Here $\Delta_j \left(j=\overline{1,n}\right)$ are the determinants formed by replacing the *j*-th column of the coefficient matrix with the column of constant terms B.

Note that, if $\Delta=0$ and at least one determinant $\Delta_i \neq 0$, then the system is *incompatible* (no solutions). If $\Delta=0$ and all $\Delta_i=0$, then the system is *indefinite* (some solutions).

Example. Solve a system of equations by using the Cramer's rule

$$\begin{cases} x_1+\ \ x_2+x_3=-2, \\ 4x_1+2x_2+x_3=-4, \\ 9x_1+3x_2+x_3=-8, \end{cases}$$

Solution.

Determinant of the system matrix is

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix}.$$

Let calculate the determinant of the system expanding it by the first row

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix}=1\cdot(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}+1\cdot(-1)^{1+2}\begin{vmatrix} 4 & 1 \\ 9 & 1 \end{vmatrix}+1\cdot(-1)^{1+3}\begin{vmatrix} 4 & 2 \\ 9 & 3 \end{vmatrix}=$$

$$=(2-3)-(4-9)+(12-18)=-1+5-6=-2\neq 0.$$

Thus, the system has a unique solution.

Calculate the determinants Δ_1, Δ_2 and Δ_3.

$$\Delta_1=\begin{vmatrix} -2 & 1 & 1 \\ -4 & 2 & 1 \\ -8 & 3 & 1 \end{vmatrix}=-2\cdot(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}+1\cdot(-1)^{1+2}\begin{vmatrix} -4 & 1 \\ -8 & 1 \end{vmatrix}+1\cdot(-1)^{1+3}\begin{vmatrix} -4 & 2 \\ -8 & 3 \end{vmatrix}=$$

$$=-2\cdot(2-3)-(-4+8)+(-12+16)=2-4+4=2,$$

$$x_1=\frac{\Delta_1}{\Delta}=\frac{2}{-2}=-1,$$

$$\Delta_2 = \begin{vmatrix} 1 & -2 & 1 \\ 4 & -4 & 1 \\ 9 & -8 & 1 \end{vmatrix} = 1\cdot(-1)^{1+1}\begin{vmatrix} -4 & 1 \\ -8 & 1 \end{vmatrix} - 2\cdot(-1)^{1+2}\begin{vmatrix} 4 & 1 \\ 9 & 1 \end{vmatrix} + 1\cdot(-1)^{1+3}\begin{vmatrix} 4 & -4 \\ 9 & -8 \end{vmatrix} =$$

$$= (-4+8) + 2\cdot(4-9) + (-32+36) = 4 + 2\cdot(-5) + 4 = 8 - 10 = -2,$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{-2}{-2} = 1,$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & -2 \\ 4 & 2 & -4 \\ 9 & 3 & -8 \end{vmatrix} = 1\cdot(-1)^{1+1}\begin{vmatrix} 2 & -4 \\ 3 & -8 \end{vmatrix} + 1\cdot(-1)^{1+2}\begin{vmatrix} 4 & -4 \\ 9 & -8 \end{vmatrix} - 2\cdot(-1)^{1+3}\begin{vmatrix} 4 & 2 \\ 9 & 3 \end{vmatrix} =$$

$$= (-16+12) - (-32+36) - 2\cdot(12-18) = -4 - 4 - 2\cdot(-6) = -8 + 12 = 4,$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{4}{-2} = -2.$$

Check up the solution of system. Let substitute the solution to the system

$$\begin{cases} -1 + \ 1 - \ 2 = -2, \\ 4\cdot(-1) + 2\cdot 1 - 2 = -4, \\ 9\cdot(-1) + 3\cdot 1 - 2 = -8, \end{cases} \Rightarrow \begin{cases} -2 = -2, \\ -4 = -4, \\ -8 = -8. \end{cases}$$

Answer is $x_1 = -1, x_2 = 1, x_3 = -2$ or $X = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$.

3.4 The Rank of Matrix

For solving of arbitrary systems of m equations with n unknowns we will introduce notation about a ***rank of a matrix***.

Let be given an arbitrary matrix A dimension of which is $m \times n$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Let us strike out k rows and k columns in this matrix. Then elements a_{ij} found at the intersection of these rows and columns form the matrix of order k.

Definition. Determinant of this matrix is called the ***minor of the k order of the matrix A***.

Definition. The highest order of the minor of matrix A different from zero is called the ***rank of this matrix*** and denoted $rang\ A$ or $r(A)$.

Definition. Matrix A is ***equivalent*** to matrix B, if their ranks are equal

$$A \sim B \Leftrightarrow r(A) = r(B).$$

Example. Find the rank of the matrix A

$$A_{3\times4} = \begin{pmatrix} 4 & 1 & -1 & 3 \\ 3 & -2 & -2 & -6 \\ 11 & 0 & -4 & 0 \end{pmatrix}.$$

Solution.

Let's select minors of 1st order, 2nd order and 3rd order

$$M_1 = 4 \neq 0,\ M_2 = \begin{vmatrix} 4 & -1 \\ 3 & -2 \end{vmatrix} = -11 \neq 0,$$

$$M_3' = \begin{vmatrix} 4 & 1 & -1 \\ 3 & -2 & -2 \\ 11 & 0 & -4 \end{vmatrix} = 0;\ M_3'' = \begin{vmatrix} 4 & 1 & 3 \\ 3 & -2 & -6 \\ 11 & 0 & 0 \end{vmatrix} = 0 \Rightarrow r(A) = 2.$$

Thus, rank of the matrix is 2.

Consider elementary operations with matrices.

Elementary Operations with Matrices

(a) Deleting any row (column) all elements of which are zeros.

(b) Interchanging any two rows (columns).

(c) Multiplying all elements in a row (column) by the same nonzero number.

(d) Adding to elements of row (column) of corresponding elements of other row (column) multiplied by a number $\lambda \neq 0$.

Theorem. Elementary operations do not change the rank of a matrix.

Remark. To find the rank of a matrix reduce it to a triangular form. A number of nonzero rows is the rank of the given matrix.

3.5 System of Linear Equations in the General Case

Let be given a system of *m* equations with *n* variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}, \tag{7}$$

Then the matrix of the system (1.7) is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Let join to the matrix of the system the column of constant terms. The matrix $\overline{A}$ is called ***the augmented matrix of the system*** (1.7)

$$\overline{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right). \tag{8}$$

If the system (7) has at least one solution, it is called a ***compatible system***.

Theorem (***Kronecker-Kapelli Theorem***). A system of linear equations (8) is compatible if and only if the rank of matrix A equals the rank of the augmented matrix $\overline{A}$

$$r(A) = r(\overline{A}).$$

Finding the set of all solutions is solving the system. We don't need guesswork or good luck; there is an algorithm that always works. This algorithm is Gauss's Method (or Gaussian elimination or linear elimination).

3.6 Gauss's Method

The aim of Gauss's Method is to rewrite the augmented matrix (1.8) in triangular form by using elementary operations.

$$\overline{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \sim \ldots \sim \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} & \tilde{b}_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \tilde{a}_{mn} & \tilde{b}_m \end{array}\right).$$

Example. Solve the system by using Gauss's Method

$$\begin{cases} x_1 + x_2 - x_3 + x_4 = -4 \\ x_1 - x_2 + x_3 + 2x_4 = 0 \\ 3x_1 + 2x_2 - x_3 + 5x_4 = -8 \\ 2x_1 - 3x_2 - x_3 + 3x_4 = -15 \end{cases}.$$

Solution.

Reduce the augmented matrix to triangular form. Let put at the first place an equation that has a first unknown coefficient equal to 1. If there is no such an equation then we divide the first equation by x_1 coefficient.

$$\overline{A}=\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & -4 \\ 1 & -1 & 1 & 2 & 0 \\ 3 & 2 & -1 & 5 & -8 \\ 2 & -3 & -1 & 3 & -15 \end{array}\right)\sim\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & -4 \\ 0 & -2 & 2 & 1 & 4 \\ 0 & -1 & 2 & 2 & 4 \\ 0 & -5 & 1 & 1 & -7 \end{array}\right)\sim\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & -1/2 & -2 \\ 0 & -1 & 2 & 2 & 4 \\ 0 & -1 & 1/5 & 1/5 & -7/5 \end{array}\right)\sim$$

$$\sim\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & -1/2 & -2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 4/5 & 3/10 & 17/5 \end{array}\right)\sim\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & -1/2 & -2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & -9/10 & 9/5 \end{array}\right)\sim\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & -1/2 & -2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right).$$

We have obtained the triangular matrix. Rank of matrix of the system equals the rank of the augmented matrix

$$r(A)=r(\overline{A})=3.$$

This means that the system has a solution. Take the forth row $x_4=-2$.

Knowing x_4 we get x_3 from the third row, x_2 from the second row x_2 and from the first row x_1

$$\begin{cases} x_1+x_2-x_3+x_4=-4, \\ x_2-x_3-1/2\,x_4=-2, \\ x_3+3/2\,x_4=2, \\ x_4=-2. \end{cases}$$

The solution of the system is $x_4=-2$; $x_3=5$; $x_2=2$; $x_1=1$.

Task “Matrices”

1. Matrices A and B are given, $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

Calculate matrices $\alpha \cdot A, \beta \cdot B, \alpha \cdot A \pm \beta \cdot B$. Values $a_{ij}, b_{ij}, \alpha, \beta$ are given in Table 1.

Table 1

Number	a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}	a_{31}	a_{32}	a_{33}	b_{11}	b_{12}	b_{13}	b_{21}	b_{22}	b_{23}	b_{31}	b_{32}	b_{33}	α	β
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**
1	2	1	3	5	2	7	4	1	3	1	3	7	5	6	9	2	1	3	4	2
2	1	7	3	5	2	6	1	4	2	8	1	3	4	9	1	5	2	4	3	2
3	4	1	3	5	6	2	1	4	3	6	1	2	5	4	3	7	3	1	5	3
4	5	2	3	4	1	2	2	5	7	3	6	1	2	5	4	3	7	6	2	7
5	6	5	2	1	6	3	4	2	1	1	2	7	6	1	3	9	5	3	6	3
6	4	2	6	1	3	7	5	2	1	2	4	6	3	2	0	5	1	8	2	4
7	1	4	3	5	2	6	0	2	1	4	3	1	2	6	5	10	11	2	3	5
8	2	4	3	5	1	2	3	7	1	5	3	2	4	6	2	0	3	3	7	6
9	3	2	1	5	4	-2	1	8	3	3	5	4	6	2	1	3	4	5	2	5
10	4	3	1	2	6	3	1	9	3	0	4	3	6	5	1	8	1	2	3	9
11	5	1	3	4	2	1	3	1	9	4	0	3	5	2	1	1	6	3	2	6
12	10	3	1	2	4	5	1	3	2	5	3	0	2	1	6	4	3	2	3	4
13	2	1	3	5	2	7	4	1	3	-8	3	7	5	6	9	2	1	3	4	2
14	1	7	3	5	2	6	1	4	2	8	1	3	4	-9	1	5	2	4	3	2
15	3	1	3	5	6	2	1	4	3	6	1	2	5	4	3	7	3	1	5	3
16	5	2	3	4	1	2	2	5	7	3	6	1	2	5	4	3	7	6	2	7
17	4	2	6	1	3	7	5	2	1	2	4	6	3	2	0	5	1	8	2	4
18	10	4	3	5	2	6	0	2	-2	4	3	1	2	6	5	10	11	2	3	5
19	-1	4	3	5	1	2	3	7	1	5	3	2	4	6	2	0	3	3	7	6
20	5	2	1	5	4	2	1	8	3	3	5	4	6	2	1	3	4	5	2	5

2. Matrices A and B are given, $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$, $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$.

Find matrices $C=A\cdot B$ and $D=B\cdot A$. Values a_{ij}, b_{ij} are given in Table 2.

Table 2

Number	a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}	b_{11}	b_{12}	b_{21}	b_{22}	b_{31}	b_{32}
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1	1	4	3	5	1	2	2	1	5	6	0	3
2	3	2	1	2	3	2	4	3	1	0	2	1
3	1	3	4	2	5	1	6	2	1	3	7	1
4	4	1	2	3	2	7	5	3	1	2	4	6
5	5	2	1	4	3	2	6	2	3	1	0	4
6	-1	6	1	2	3	4	1	2	7	3	4	1
7	2	-3	1	4	6	3	3	2	1	5	4	3
8	3	2	3	4	5	6	2	4	1	2	3	0
9	-2	3	1	3	4	6	5	2	3	4	1	3
10	6	1	2	3	4	1	7	4	5	3	1	2
11	-4	2	1	6	3	2	4	2	3	-2	1	5
12	3	-5	1	4	2	3	-1	4	2	1	3	5
13	3	2	3	4	5	6	2	4	1	2	3	0
14	-2	3	1	3	4	6	5	2	3	4	1	3
15	6	1	2	3	4	1	7	4	5	3	1	2
16	-4	2	1	6	3	2	4	2	3	-2	1	5
17	3	-5	1	4	2	3	-1	4	2	1	3	5
18	6	-1	7	2	3	4	1	4	7	6	2	1
19	5	-3	2	4	6	1	2	-3	4	2	1	3
20	10	-1	2	3	5	1	3	4	1	5	3	6

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