ON FAILURE INTENSITY FUNCTION OF NONHOMOGENEOUS POISSON PROCESS

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Overwhelming majority of publications on Nonhomogeneous Poisson process considers just two monotonic forms of the NHPP's ROCOF: the log−linear model the power law model. In this paper, we propose to capitalize on the fact that NHPP's rate of occurrence of failures (ROCOF) formally coincides with the hazard function of the underlying life time distribution. Therefore, the variety of parametric forms for the hazard functions of traditional life time distributions could be used as the models for the ROCOF of respective NHPPs. Moreover, the hazard function of a mixture of underlying distributions could be used to model the non−monotonic ROCOF. Parameter estimation of such ROCOF models reduces to the estimation of the cumulative hazard function of the underlying life time distribution. We use real−world automotive data to illustrate the point.

1. Introduction

The Nonhomogeneous Poisson Process (NHPP) is widely used to model the failure process of repairable systems. A key assumption of the NHPP model, in reliability context, is that upon a failure, the system is repaired to the condition, as it was right before the failure, a.k.a., the *minimal repair* or *same−as−old* repair assumption. This assumption seems quite appropriate for a repairable system such as an automobile, since typically only a small part (i.e., component or a subsystem) of an automobile is repaired at a time thus restoring it back to the condition close to the same as it was before the failure.

Like any other point process, the NHPP is characterized by the *cumulative intensity function* (CIF), $\Lambda(t)$, which is the expected number of failures as a function of operating time, t

$$\Lambda(t) = \int_0^t \lambda(\tau)d\tau, \qquad (1)$$

where $\lambda(\tau)$ is known as the *rate of occurrence of failures* or ROCOF.

An overwhelming majority of publications on the reliability applications of the NHPP, including Ascher and Feingold [1], Thompson [2], Crowder et al. [3], Hoyland and Rausand [4], consider just two monotonic forms of the NHPP ROCOF. The first one is the *log−linear model* discussed by Cox and Lewis [5], and the other one is the *power law model* discussed by Crow [6]. Other, more complex forms of ROCOF considered in the literature are either a combination of the above two (e.g., [7]) or generalizations of the log−linear model to include polynomial terms in t (e.g., [8]).

In this paper, we would like to discuss another family of the ROCOF models, which extend the practical application of the NHPP in reliability engineering.

2. NHPP and its Underlying Distribution

For the sake of this discussion, it is convenient to think of the NHPP as a *renewal* process with "same−as−old" type of renewal upon each failure. Such a renewal process is called a *generalized renewal* (or *G−renewal*) *process* and the respective renewal equation is given by Kijima et. al [9]

$$\Lambda(t) = \int_0^t \left(g(\tau \mid 0) + \int_0^\tau h(x) g(\tau - x \mid x) dx \right) d\tau, \quad (2)$$

where

$$g(t \mid x) = \frac{f(t+qx)}{1-F(qx)}, \quad t, x \ge 0,$$

is defined such that $g(t|0) = f(t)$, and $F(.), f(.), h(.)$ are the CDF, PDF and the hazard function of the time to first failure distribution of the *underlying distribution*; and q is the so−called *restoration* (or *repair effectiveness*) *factor*.

By setting $q = 1$, the GRP reduces to the NHPP, and the CIF of GRP reduces to that of NHPP

$$\begin{aligned} \Lambda(t) &= \int_0^t \left(f(\tau) + \int_0^\tau h(x) \frac{f(\tau - x + x)}{1-F(x)} dx \right) d\tau = \\ &= \int_0^t \left(f(\tau) + f(\tau) \int_0^\tau \frac{h(x)}{1-F(x)} dx \right) d\tau = \\ &= \int_0^t \left(f(\tau) + f(\tau) \left(\frac{1}{1-F(x)} \Big|_0^\tau \right) \right) d\tau = \\ &= \int_0^t \left(f(\tau) + h(\tau) - f(\tau) \right) d\tau = H(t). \end{aligned} \qquad (3)$$

Observing the beginning and the end of the above equation, one can infer some useful properties:

Property 1. *The cumulative intensity function of the NHPP is formally equal to the cumulative hazard function of the underlying failure time distribution.*

It is obvious that the same is true with respect to the derivatives of these functions:

Property 2. *The ROCOF of the NHPP is formally equal to the hazard function of the underlying failure time distribution.*

These properties of the NHPP allow using the hazard functions of known failure time distributions as the models for the NHPP ROCOF. Thus, besides the traditionally used Weibull as the underlying distribution (in which case the respective ROCOF is the same as the power law model mentioned above), we propose to promote the use of other popular in life data analysis distributions including lognormal, normal, Gumbel, etc. Moreover, the hazard function of a (finite) mixture of underlying distributions could be used to model the non−monotonic ROCOF trends.

3. Statistical Estimation of NHPP

Let the number of repairs, *N(t)*, to occur in (0, *t*] be given by $N(t) = \max\{k \mid T_k \le t\}$ for *k* = 1, 2, …

The CIF in (0, *t*] is then given by $\Lambda(t) = E[N(t)]$. A natural estimator of $\Lambda(t)$[3] is

$$\hat{\Lambda}(t) = \frac{1}{N}\sum_{j=1}^{N} N_j(t), \tag{4}$$

where *N* – is a number of systems,

$\sum_{j=1}^{N} N_j(t)$ – is the total observed number of repairs in (0, *t*].

If the NHPP is the governing failure process, then due to Property 1, the above nonparametric estimator can be used both for the CIF of the point process *and* the cumulative hazard function (CHF) of the underlying distribution

$$\hat{\Lambda}(t) = \hat{H}(t) = \frac{1}{N}\sum_{j=1}^{N} N_j(t), \tag{5}$$

Once the nonparametric estimate of the CHF is obtained, one can use standard statistical procedures, e.g., hazard papers [10], for the estimation of the underlying distribution parameters. These parametric estimates will be the same as those to be used in the respective parametric model of NHPP ROCOF.

4. Numerical Example

Figures 1 and 2 show cumulative intensity functions estimated for two separate (repairable) automotive subsystems. Figure 1 juxtaposes the empirical cumulative intensity function modeled via the traditional, "Weibull−based", NHPP versus that modeled via "log-normal−based" NHPP. One could observe the lack of fit in the left part of the figure, which is rectified in the right part of the figure.

Figure 2 juxtaposes the empirical cumulative intensity function modeled via the traditional, "Weibull−based" NHPP versus that modeled via "Weibull−mixture−based" NHPP. As evidenced from the figure, the Weibull−mixture−based NHPP models the non−monotonic trend of the empirical ROCOF more closely than the traditional power law NHPP, which is constrained to the monotonic (in this case deteriorating) trend. It must be noted that one has to exercise caution *extrapolating* Weibull−mixture−based CIF.

The estimation in both cases was done using the least squares (hazard paper) procedure with subsequent nonlinear least squares refinement of the estimates.

References

1. Ascher H., Feingold H. *Repairable Systems Reliability*, New York: Marcel Dekker, 1984.
2. Thompson WA, Jr. *Point Process Models with Applications to Safety and Reliability*. London: Chapman and Hall, 1988.
3. Crowder MJ, Kimber AC, Smith RL, and Sweeting TJ. *Statistical Analysis of Reliability Data*, London: Chapman and Hall, 1991.
4. Rausand M and Høyland A. System Reliability Theory: Models, Statistical Methods, and Applications, New York: Wiley, 1994.
5. Cox DR and Lewis PA. *The Statistical Analysis of Series of Events*, London: Methuen, 1966.
6. Crow LH. Reliability Analysis for Complex Repairable Systems. In: *Reliability and Biometry*, F. Proschan and R.J. Serfling, eds, SIAM, 1974, p. 379−410.
7. Lee L. Testing Adequacy of the Weibull and Log Linear Rate Models for a Poisson Process. *Technometrics*, 1980, Vol. 22, No. 2, p. 195−199.
8. MacLean CJ. Estimation and Testing of an Exponential Polynomial Rate Function within the Non−stationary Poisson Process. *Biometrca*, 1974, 61, p. 81−85.
9. Kijima M and Sumita N. A Useful Generalization of Renewal Theory: Counting Process Governed by Non−negative Markovian Increments. *J. Appl. Prob.*, 23, 1986, p. 71−88.
10. Nelson W. *Applied Life Data Analysis*, New York: John Wiley & Sons, 1982.

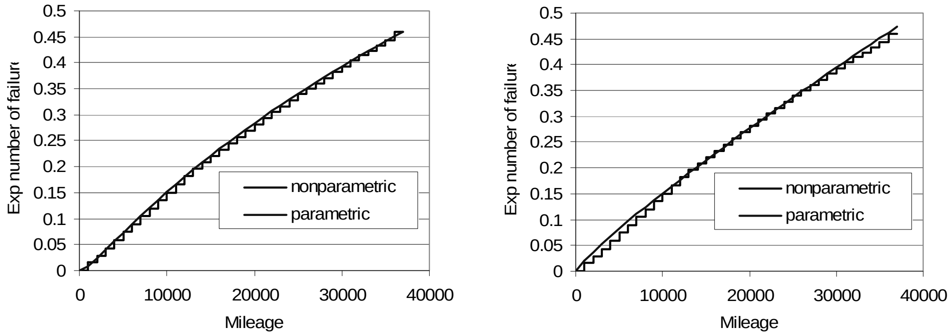

Figure 1 – Empirical cumulative intensity function modeled via Weibull−based NHPP (above) and lognormal−based NNHP (below)

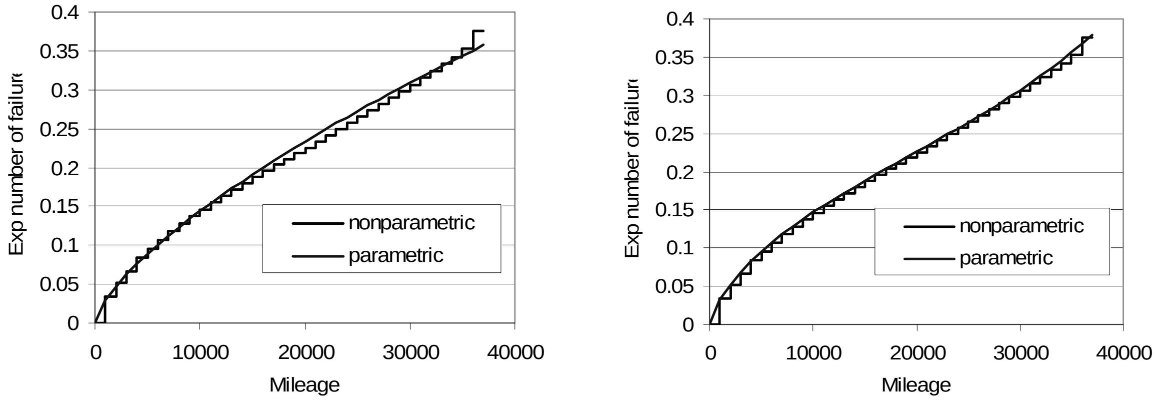

Figure 2 – Empirical cumulative intensity function modeled via Weibull−based NHPP (above) and Weibull−mixture−based NNHP (below)

Аннотация

К ВОПРОСУ О ВЫБОРЕ МОДЕЛИ ФУНКЦИИ ПОТОКА ОТКАЗОВ НЕОДНОРОДНОГО ПУАССОНОВСКОГО ПРОЦЕССА

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Подавляющее число публикаций по неоднородному пуассоновскому процессу (НПП) используют лишь две монотонные параметрические модели функции потока отказов: логлинейную и степенную. В данной работе показано, что функция потока отказов НПП формально совпадает с функцией интенсивности базового распределения, формирующего данный НПП. В этой связи, все разнообразие параметрических форм функции интенсивности распределений, используемых в теории надежности, может быть использовано в качестве модели функции потока отказов НПП. Более того, функция интенсивности смеси базовых распределений может быть использована для моделирования немонотонной функции потока отказов НПП. Оценка параметров НПП сводится к оценке параметров базового распределения НПП. Суть предлагаемого подхода проиллюстрирована на данных по надежности из автомобильной промышленности.

Анотація

ДО ПИТАННЯ ПРО ВИБІР МОДЕЛІ ФУНКЦІЇ ПОТОКУ ВІДМОВ НЕОДНОРІДНОГО ПУАССОНІВСЬКОГО ПРОЦЕСУ

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Переважна кількість публікацій по неоднорідному пуассонівському процесу (НПП) використовують лише дві монотонні параметричні моделі функції потоку відмов: логлінійну і степеневу. В даній роботі показано, що функція потоку відмов НПП формально збігається з функцією інтенсивності базового розподілу, що формує даний НПП. У зв'язку з цим, все розмаїття параметричних форм функції розподілів інтенсивності, що використовуються в теорії надійності, може бути використано в якості моделі функції потоку відмов НПП. Більш того, функція інтенсивності суміші базових розподілів може бути використана для моделювання немонотонної функції потоку відмов НПП. Оцінка параметрів НПП зводиться до оцінки параметрів базового розподілу НПП. Суть пропонованого підходу проілюстровано на даних з надійності в автомобільній промисловості.