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REGULARIZATION METHODS FOR KIJIMA MODELS

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The most comprehensive prediction model is the g–renewal process proposed by Kijima [1], which allows for modelling of both perfect and imperfect repairs through the use of the so–called restoration factor. Krivtsov and Yevkin [2] showed that statistical estimation of the g–renewal process parameters is an ill–posed inverse problem (the solution is not unique and/or is sensitive to statistical errors). They proposed a regularization approach specifically suited to the g–renewal process: separating the estimation of the underlying life distribution parameters from the restoration factor in two consecutive steps. Using numerical studies, they showed that the estimation/prediction accuracy of the proposed method was considerably higher than that of the existing methods. This paper elaborates on more advanced regularization techniques, which allow to even further increase the estimation/prediction accuracy in the framework of both Least Squares and Maximum Likelihood estimation. Proposed regularization becomes especially useful for limited sample sizes. The accuracy and efficiency of the proposed approach is validated through extensive numerical studies under various underlying lifetime distributions including Weibull, Gaussian and log–normal.

Acronyms:

CDF	cumulative distribution function
CIF	cumulative intensity function
GPR	generalized renewal (g–renewal) process
MLE	maximum likelihood estimation
LSQ	least residual squares estimation
PDF	probability density function

1. Introduction. The problem of recurrent failure prediction arises in forecasting warranty repairs/cost, maintenance optimization and evaluation of repair quality. The most popular model of the g–renewal process is suggested by Kijima [1] by introducing the notion of *virtual age* defined by the *restoration* factor, q. If q=0, the repair is perfect. If q> 0,the repair is imperfect including the case of q=1, when the system is restored to the "same–as–old" condition.

We have shown in [2] that, in general case, parameter estimation of g–renewal process is an ill–posed problem. It means that the obtained solution can be non–unique and significantly depends on small changes in the input data. Typically, additional information is required to resolve an ill–posed problem. It is suggested in [2] to solve the problem in two steps. At the first step, only the underlying life time distribution is estimated using the time to *first* failures only. At the second step, only restoration parameter q is estimated using *all* recurrent failure times. Obviously, this approach not only converts an ill–posed problem into a regular one, but is also very efficient in terms of computational time, because the estimated parameters are decoupled and at the second step (most time consuming) only one parameter is estimated. This approach works well, if at the first step, parameters of the underlying failure time distribution are estimated with good accuracy based on a relatively large sample size of observed failure times. In this paper, we suggest an improvement to this approach and show its efficiency for small sample sizes, which is often a restriction in practical applications.

2. Background of the problem. Let us consider the cumulative *intensity function* (CIF) $W(t)$ corresponding to an arbitrary underlying *cumulative distribution function* (CDF)and the restoration factor $q \neq 1$. We can always make q=1 and find the exact solution for underlying CDF in the closed form $F(t) = 1 - e^{-W(t)}$, which would differ from the given one simply because q is different. This example shows that the inverse g–renewal problem has no unique solution in general case. Another obvious example is the case when the underlying failure times follow the exponential distribution. In this particular case, the restoration factor q cannot be estimated, because with the underlying exponential distribution, the g–renewal process does not depend on q.

As most inverse problems, the g–renewal inverse problem is ill–posed. It means that a numerical algorithm can converge to different solutions depending on selected initial values for the parameters, even if the type of underlying CDF (e.g., Weibull) is specified. Example 1 in the Appendix shows failure times for 5 repairable systems, which were observed until time $T=40$ (suspension time). The failure times were generated by the Monte Carlo method using Kijima I model with the restoration factor q=1 and the underlying log–normal distribution with parameters: $\mu = 2, \sigma = 0.5$.

The *maximum likelihood estimation* (MLE) method yields two sets of solutions:

$$\hat{\mu}_1 = 1.094, \hat{\sigma}_1 = 0.924, \hat{q}_1 = 0$$
$$\hat{\mu}_2 = 1.751, \hat{\sigma}_2 = 0.569, \hat{q}_2 = 0.577$$

both corresponding to the two local maximums of the log–likelihood function $L_1 = -105.86, L_2 = -107.86$. These maximal values of the log–likelihood are close to each other and are obtained by *the same* Newton–Raphson method but with different initial values for restoration factor: q=0.1 and q=0.5. We will consider this example in more details in Section 4 but, for now, one must admit that the algorithm can converge to different solutions, especially in presence of significant statistical noise.

To avoid this uncertainty, the following regularization method is suggested in [2] for ill–posed inverse g–

renewal problem. At the first step, only times to first failures are considered and parameters of the underlying CDF are estimated. Considering these parameters as known, the restoration factor q is then estimated at the second step. It is an efficient approach, if the CDF parameters have been estimated with a good accuracy based on sufficient sample size of first failure times (i.e., the number of the repairable systems under observation is sufficiently large).

However, for a smaller number of systems under observation, the estimation error of the underlying CDF can be quite significant. Notably, times to subsequent failures (beyond the first failure)depend not only on the restoration factor, but also on the underlying CDF parameters. So, these subsequent failure times can be used to even better estimate the CDF of the times to the first failures. (It's understood that with q=0,when upon a failure a system is replaced by a new one, all times between subsequent failures can be considered as times to first failures.)

In this paper, we suggest the following improvement to our algorithm originally proposed in [2]. The first step remains the same; however, the obtained estimates of the underlying CDF are now used as the initial values for the second step, whereat all model parameters are estimated simultaneously. This approach allows, on the one hand, avoiding irrelevant solutions, and on the other hand, improving computational speed, as the number of iterations significantly depends on the selected initial values.

In the remainder, we will describe the proposed approach and the computational algorithm (Section 3) using both the MLE and the least residual squares (LSQ) methods, consider several examples illustrating the algorithm (Section 4), and compare the MLE and LSQ methods (Section 5).

3. Proposed methodology and the algorithm. For sake of simplicity we assume that time between failures is much greater than the time to repair in the g–renewal process. Let t_1 be time to the first failure, t_2 be the time between the first failure and the second failure, so that t_i is the time between the (i–1)–th and i–th failures. According to Kijima and Sumita [1], the probability to next i–th failure is defined as:

$$F_i(t_i) = \frac{F(A_{i-1} + t_i) - F(A_{i-1})}{1 - F(A_{i-1})} \tag{1}$$

where $F(t)$ is underlying CDF and A_i is the virtual age that depends on the restoration factor q.

Two models of aging are suggested in [1]. Model I:

$$A_i = q\tau_i \tag{2}$$

where τ_i is the real age at the i th failure, A_0=0. According to Model II :

$$A_i = qA_{i-1} \tag{3}$$

if i > 1.The corresponding *probability density function* (PDF) is the derivative of the cumulative distribution function (1):

$$f_i(t_i) = \frac{f(A_{i-1} + t_i))}{1 - F(A_{i-1})} \tag{4}$$

where $f(t)$ is the PDF of underlying failure time distribution.

In case of N repairable systems, for the j–th system we will use the following notation $F_i^j(t_i^j)$ and $f_i^j(t_i^j)$ instead of (1) and (4).

3.1. LSQ method. The rank regression method is used in this case and the residual squares function is constructed. Considering first failures only, the sum of residual squares is:

$$S_1 = \sum_{j=1}^{N} \left(F_1^j(t_1^j) - \widetilde{F}^j(t_1^j)\right)^2 \tag{5}$$

where $\widetilde{F}^j(t_1^j)$ are the nonparametric estimates of the time to first failures CDF obtained via a rank regression method.

Equation (5)is minimized with respect to the underlying CDF parameters.

A similar expression can be constructed for the CIF residual squares (that now includes all failures, not only the first ones):

$$S_2 = \sum_{j=1}^{N} \left(W_1^j(t_1^j) - \widetilde{W}^j(t_1^j)\right)^2 \tag{6}$$

which additionally depends on the restoration factor, q.

Mathematically, we have to find the minimum of S_2 under the condition that S_1 is also minimal; therefore, it is reasonable to consider the minimum of the following functions

$$S = S_1 + S_2 \tag{7}$$

instead of (6). At the first stage, we solve for the parameters of the underlying CDF by minimizing (5). This is the easiest part of the calculation, because the minimum is available in a simple form for most underlying CDF's. To minimize (7) with respect to all model parameters (including q),as the next step, we apply the advanced Monte Carlo method discussed in [3]. It allows to efficiently calculate the CIF for the given system parameters and even partial derivatives of the CIF with respect to system parameters. The values of derivatives are used in finding the minimum of (7) by applying the Gauss–Newton algorithm, which is efficient in most non–linear regression problems. The standard error, SE, corresponding to (7) is calculated as

$$SE = \sqrt{S/(r_1 + r_2)} \tag{8}$$

where r_1 and r_2 is the number of first failures (as in (5)) and all failures (as in (6)), respectively.

3.2. MLE method. The log–likelihood function for the CIF can be represented as:

$$L = \sum_{j=1}^{N}\left(\sum_{i=1}^{n_j}\ln\left(f_i^j(t_i^j)\right)+\ln\left(1-F(T_j)\right)\right) \tag{9}$$

The second term inside the internal sum corresponds to the suspension, if the suspension time exceeds the time of the last failure of the j–th item with T_j being the difference between these two times. Equation (9) can be explicitly written in terms of the underlying distribution parameters and q– for the case of Weibull, Gaussian and log–normal distributions. It is also possible to take first partial derivatives of (9)with respect to all g–renewal model parameters. For the case of the Weibull distribution, the solution can be found in [4]. We use the Newton–Raphson iteration method to find the maximum of (9). Second partial derivatives are calculated numerically.

We can change the order of summation and regroup (9) as

$$L = \sum_{i=1}^{n} L_i \tag{10}$$

where L_i accumulates all terms (including the suspension terms, if any) of all systems corresponding to the i–th failure.

Especially, we are interested in first term corresponding to first failures. It is obvious that equations (1) and (4) yield the CDF of the underlying failure time distribution, because A_0=0, if $i=1$, and virtual age of the system equals its real age at the first failure. Therefore,L_1 is exactly the log–likelihood used in the time to first failures estimation and is thus already included in (9).It is remarkable that we do not need to include L_1 in (9) as a separate term.

3.3. The algorithm. The prosed estimation procedure is as follows:

1. Using the rank regression method, estimate the underlying CDF parameters taking into account only first failures by minimizing (5).
2. The obtained estimates are used as initial values in the MLE method to further refine the underlying CDF parameters. The corresponding log–likelihood function L_1 is maximized using the Newton–Raphson algorithm.
3. The estimated CDF parameters are used as the initial in this step. The Newton–Raphson iteration algorithm is used to maximize the likelihood function (9) with respect to *all*g–renewal model parameters. We recommend repeating this calculation step for several initial values of the restoration factor.
4. The LSQ method is applied in this step. Previously obtained result is used as initial values for the Gauss–Newton iteration method minimizing (7).
5. Confidence intervals are estimated using the Fisher information matrix and the numerically calculated partial derivatives of the CIF with respect to all model parameters. Advanced Monte Carlo method is applied here according to [4].

Steps 1–3 take approximately 1 second to calculate using a laptop with a 1.7GHz of processor speed and 8.0GB of RAM. Step 4 takes several more seconds. We recommend to still perform this step, especially if the MLE in Step 3 converges to different values depending on initial value of the restoration factor (see Example 1). The most time consuming is Step 5 – it takes several seconds. The algorithm is implemented for Kijima model I and II in the free online calculation software[6]. In addition, the mean time to repair can be introduced (with some restrictions) as a calculation option.

4. Examples. We will consider 3 examples illustrating some specific features of the proposed procedure and the efficiency of the algorithm. Input data were generated by Monte Carlo simulation [5] and are given in the Appendix.

4.1. Example 1. This example shows two solutions corresponding to two different initial values of the restoration factor. The failure times (Example 1 in the Appendix) were simulated by using Kijima I model with the restoration factor of q=1 and the log–normal underlying failure time distribution with the following parameters: $\mu = 2, \sigma = 0.5$. Five repairable systems were observed until T=40 (suspension time).

Below is the summary of the estimation results (corresponding to the first 4 steps of the proposed procedure) with the initial value of the restoration factor q=0.1:

1. The rank regression method yields the following estimates for the underlying CDF parameters: $\hat{\mu} = 1.86,\ \hat{\sigma} = 0.908$.
2. The MLE method shows a somewhat better (closer to the original values) estimation of the CDF parameters: $\hat{\mu} = 1.86,\ \hat{\sigma} = 0.693$.
3. The MLE method for Kijima model I yields the maximum log–likelihood: $L = -105.96$ corresponding to parameters $\hat{\mu} = 1.094, \hat{\sigma} = 0.924, \hat{q} = 0$ (see Fig. 1)
4. The LSQ solution is $\hat{\mu} = 1.20, \hat{\sigma} = 0.846, \hat{q} = 0.021$ with the standard error of0.379 (for the entire model).

The obtained MLE solution for the CIF is depicted in Fig. 1 with the 90% two–sided confidence interval.

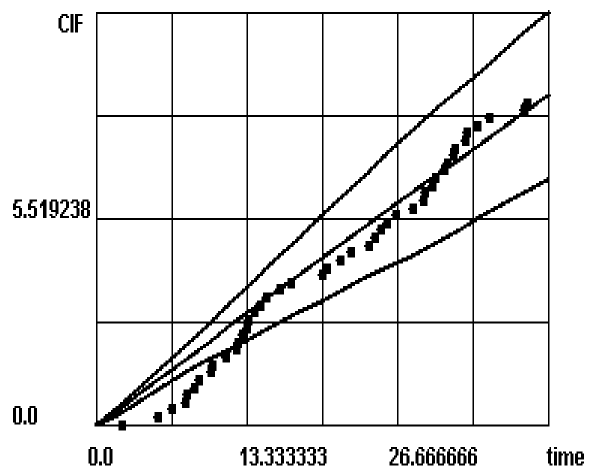

Fig. 1 – MLE solution ($\hat{\mu} = 1.094, \hat{\sigma} = 0.924, \hat{q} = 0$) with 90% two sided confidence interval.

Below is the summary of the estimation results with the initial value of the restoration factor q=0.5:

1. The rank regression method for the underlying CDF yields: $\hat{\mu}=1.86,\ \hat{\sigma}=0.908$.
2. The MLE method yields a somewhat better estimation: $\hat{\mu}=1.86,\ \hat{\sigma}=0.693$.
3. The MLE method for the Kijima model yields the maximum log–likelihood $L=-107.86$ corresponding to parameters $\hat{\mu}=1.75, \hat{\sigma}=0.569, \hat{q}=0.577$.
4. The LSQ solution is $\hat{\mu}=1.70, \hat{\sigma}=0.572\ \hat{q}=0.492$ with the standard error of0.280.

The obtained MLE solution for the CIF is depicted in Fig. 1 with the 90% two–sided confidence interval.

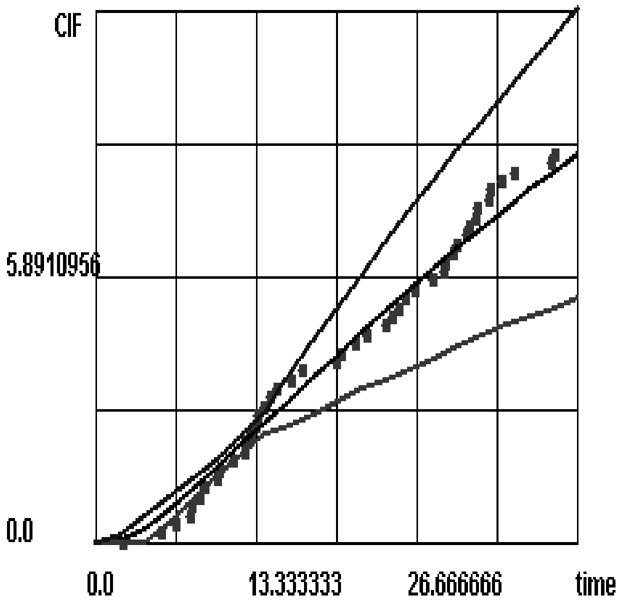

Fig. 2 – MLE solution ($\hat{\mu}=1.75, \hat{\sigma}=0.569, \hat{q}=0.577$) with 90% two–sided confidence interval.

Comparing these two results, we can conclude that there are two solutions with the local maximum log–likelihood values close to each other but with significantly different estimates of the g–renewal model parameters. The two LSQ solutions are close to the MLE solutions. However, the second LSQ solution yields a much lower standard error and the second solution better fits the non-parametric CIF. This example is rather exceptional, but it illustrates the specific nature of the problem and the importance of selecting initial values for the iteration methods not only in terms of the convergence speed, but also in terms of choosing right solution, which can be non–unique.

4.2. Example 2. The failure times (Example 2 in the Appendix) in this example were simulated by using Kijima model I with the restoration factor q=0.5 and the underlying Weibull CDF with scale parameter $\eta=1$and shape parameter $\beta=2$. Five systems were observed until the suspension time of T=2. Below is the summary of the estimation results (corresponding to the first 4 steps of the proposed procedure):

1. The rank regression method for the underlying CDF yields: $\hat{\beta}=3.422,\ \hat{\eta}=0.855$.
2. The MLE method yields the following estimates: $\hat{\beta}=4.49,\ \hat{\eta}=0.843$.
3. The MLE method applied to Kijima model yields the maximum of the log–likelihood: $L=-4.358$ corresponding to the flowing parameter estimates: $\hat{\beta}=2.76, \hat{\sigma}=0.814, \hat{q}=0.175$.
4. The LSQ solution is $\hat{\beta}=2.63, \hat{\eta}=0.816, \hat{q}=0.254$ with the overall standard error of 0.144.

The Weibull parameter estimates have improved significantly at the last two steps. The obtained LSQ solution is shown in Fig. 3 with the 90% two–sided confidence interval.

It must be noted that the CIF (and, therefore, the likelihood function as well as the residual sum) is most sensitive to the scale parameter (or the mean time to failure) of the underlying distribution, then to the shape parameter (or the standard deviation) and finally to the restoration factor. For example, relatively large changes in q can be compensated by some small changes of the scale parameter. Hence, the importance of the estimation accuracy of the respective parameters has the same order (sequence).

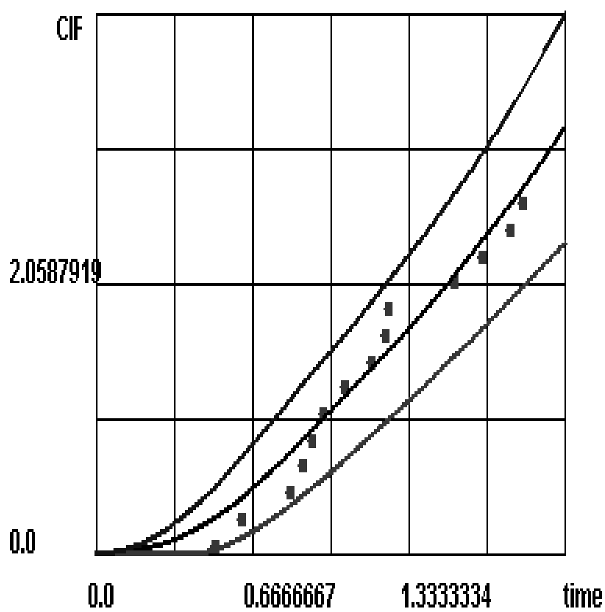

Fig. 3– LSQ solution ($\hat{\beta}=2.63, \hat{\eta}=0.816, \hat{q}=0.254$) with the 90% two–sided confidence interval.

Let us now compare the obtained CIF estimation with the result when Weibull parameters ($\hat{\beta}=3.422,\ \hat{\eta}=0.855$) are fixed after step 1 of calculation as recommended in [2]. The obtained CIF is shown in Fig. 4. The corresponding restoration factor $\hat{q}=0.315$. The curve fits better, when the CIF is less than 1, but, overall, the standard deviation is 0.167, which is larger than 0.144 obtained via the improved technique. If the MLE method is applied with fixed Weibull parameters $\hat{\beta}=4.49,\ \hat{\eta}=0.843$, we obtain solution with $\hat{q}=0.212$ and the log–likelihood of–6.84, which is much lower than –4.358 for the process parameters $\hat{\beta}=2.63, \hat{\eta}=0.816, \hat{q}=0.254$ (Fig. 3).

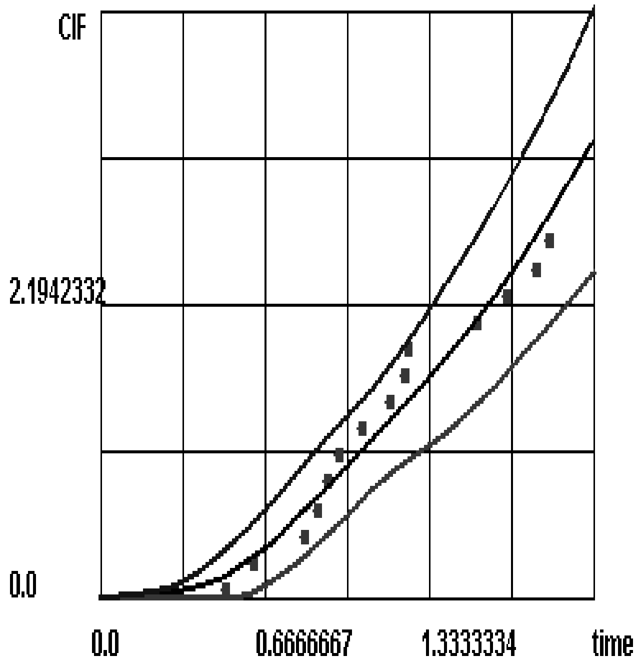

Fig. 4 – LSQ solution ($\hat{\beta}=3.42, \hat{\eta}=0.855, \hat{q}=0.315$) with the 90% two–sided confidence interval.

4.3. Example 3. The failure times (Example 3 in the Appendix) were simulated by using Kijima model I with the restoration factor q=0 and the underlying Gaussian failure time distribution function with the following mean and standard deviation: $\mu=4, \sigma=0.5$. Five systems were observed until the suspension time of T=15.

We *intentionally* estimated the system parameters using the Weibull model, i.e., as if we don't know the parametric form of the underlying CDF:

1. The rank regression method for the underlying CDF yields: $\hat{\beta}=7.47,\ \hat{\eta}=4.43$. The corresponding mean and standard deviation: $\hat{\mu}=4.158, \hat{\sigma}=0.658$.
2. The MLE method yields: $\hat{\beta}=8.63,\ \hat{\eta}=4.41$ with the corresponding mean and standard deviation $\hat{\mu}=4.168, \hat{\sigma}=0.578$.
3. The MLE yields the maximum of the log–likelihood $L=-11.83$ corresponding to $\hat{\beta}=9.24, \hat{\eta}=4.33, \hat{q}=0$ $(\hat{\mu}=4.105, \hat{\sigma}=0.532)$.

The obtained MLE solution is depicted in Fig. 5 with the 90% two–sided confidence interval. The Weibull mean and the standard deviation for the obtained CDF estimates are4.105 and 0.532, respectively. This is close to the original parameters of the Gaussian distribution $\mu=4, \sigma=0.5$. The estimation of the g–renewal process parameters with the underlying Gaussian distribution yields slightly better result for log–likelihood function: $L=-10.75$, $\hat{\mu}=4.10, \hat{\sigma}=0.464$. This example also illustrates oscillating behavior of the CIF (i.e., or, highly localized failure events), when the standard deviation of the underlying CDF is much less than its mean.

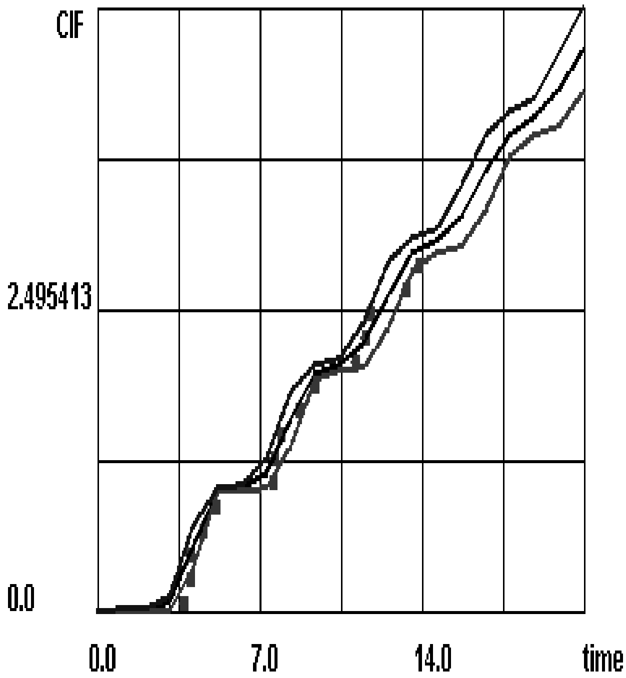

Fig. 5 – MLE solution ($\hat{\beta}=9.24, \hat{\eta}=4.33, \hat{q}=0$) with the 90% two–sided confidence interval.

Despite the fact that the considered examples were based on Kijima I model, the proposed procedure is equally applicable to Kijima II model.

5. Concluding remarks. We have conducted extensive numerical experiments using Monte Carlo simulation [5] with various types of underlying CDF's (Gaussian, log–normal and Weibull) and different values of their parameters. Three values of restoration factor 0; 0.5; 1.0 were used in the simulation. On average, 2 to 4 failures occurred before a given suspension time.

In each simulation, we observed 5 identical reparable systems. Each system failed at least once. We used the simulated data for the g–renewal model parameters estimation using both the MLE and the LSQ methods. We measured the efficiency of each method by comparing the estimated results with the exact (original) data and demonstrated the advantages of the proposed advanced regularization.

We also noticed that in the considered setting, there was not any clear preference of the MLE vs. the LSQ method or vice versa. It is probably not surprising, because in are current failure process, the suspension time plays a less significant role, as the time between the last failure and the suspension is significantly smaller, on average, than the total observation time. However, the MLE method is much faster, because it is based on a relatively simple equation for the log–likelihood function and its partial derivatives. Estimation based on the LSQ procedure can be used as a second method for validating results obtained by the MLE method. Example 1 shows that it can be used as an additional criterion to avoid irrelevant solutions in the considered ill–posed inverse problem.

APPENDIX

Example 1

1. 10.05; 12.19; 13.20; 14.83; 29.81; 30.82
2. 16.07; 28.93
3. 6.49; 8.47; 9.94; 12.80; 13.81; 19.83; 24.52; 25.53; 27.74; 28.75; 34.62
4. 2.04; 7.64; 13.27; 14.28; 16.94; 20.22; 23.96; 25.00; 29.56; 30.57; 31.58; 32.59; 33.60; 37.65
5. 5.24; 7.81; 8.89; 11.22; 12.30; 21.34; 22.35; 26.38; 31.46; 32.47; 37.95

Example 2

1. 0.870171; 0.959605; 1.638113
2. 0.494943; 0.912076; 1.164948;1.224501
3. 1.050094; 1.518585
4. 0.8146915; 1.813109
5. 0.606765; 1.236887; 1.755929

Example 3

1. 3.915579; 7.819686; 11.467476
2. 4.457433; 9.254106; 13.789127
3. 5.011945; 8.577584; 13.196124
4. 3.873146; 7.508159; 11.028036
5. 3.637726 7.467283; 11.635722

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Аннотация

МЕТОДЫ РЕГУЛЯРИЗАЦИИ МОДЕЛЕЙ КИЖИМЫ

В. Кривцов, А. Евкин

Наиболее обобщенные модели надежности восстанавливаемых систем были предложены Кижимой (1986) путем введения т.н. процесса g-восстановления. Кривцов и Евкин [2] показали, что статистическое оценивание такого рода процесса относится к классу плохо обусловленных, обратных задач (для которых решение неуникально или очень чувствительно к статистическому разбросу данных) и требует регуляризации по Тихонову. Было предложено регуляризовать эту задачу путем раздельного оценивания параметров базового распределения и параметра восстановления. В данной статье обсуждается дальнейшее усовершенствование метода Кривцова–Евкина в рамках процедур наименьших квадратов и максимального правдоподобия. Усовершенствованная регуляризация особенно эффективна для выборок малых объемов. Оценочная и экстраполяционная точность предложенного метода подтверждена численными экспериментами для ряда базовых распределений, включающих гауссовское, логнормальное и Вейбулла–Гнеденко.

Анотація

МЕТОДИ РЕГУЛЯРИЗАЦІЇ МОДЕЛЕЙ КІЖІМИ

В. Кривцов, А. Евкін

Найбільш узагальнені моделі надійності відновлюваних систем були запропоновані Кіжімой (1986) шляхом введення т.зв. процесу g-відновлення. Кривцов і Евкін [2] показали, що статистичне оцінювання такого роду процесса відноситься до класу погано обумовлених, обернених задач (для яких рішення неунікальне або дуже чутливо до статистичного розкиду даних) і вимагає регуляризації по Тихонову. Було запропоновано регуляризувати цю задачу шляхом роздільного оцінювання параметрів базового розподілу і параметра відновлення. У даній статті обговорюється подальше удосконалення методу Кривцова-Евкіна в рамках процедур найменших квадратів і максимальної правдоподібності. Удосконалена регуляризація особливо ефективна для вибірок малих обсягів. Оціночна і екстраполяція точність запропонованого методу підтверджена чисельними експериментами для ряду базових розподілів, що включають гаусовське, логнормальне і Вейбулла-Гнеденко.